# Stability analysis and simulation based on an improved Aw-Rascle model

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**Abstract.** In this paper, the improved Aw-Rascle model with viscosity term is developed to a new nonlinear dynamical system composed of ordinary differential equations, by using traveling wave substitution and a kind of variable substitution. The stability of the system is discussed by the method of phase plane analysis after getting the travelling-wave solutions, then the final data is simulated by Matlab to verify the analysis conclusion.

**Keywords:** Aw-Rascle model, Nonlinear traffic phenomenon, phase plane analysis.

## 1 Introduction

Traffic flow system has complex and stochastic characteristics, and the accuracy of traffic flow theory directly affects the prediction and planning of traffic conditions, which has attracted great attention from many researchers in recent years. Traffic flow theory analyzes the traffic phenomenon of pedestrians and vehicles on roads, establishing a mathematical model that can describe the general characteristics of actual traffic and explore the relationship among traffic flow, density and speed, finally achieving the purpose of alleviating and preventing traffic congestion fundamentally to reduce the traffic delay. The macroscopic model in the traffic flow model regards the traffic flow as a compressible continuous fluid medium composed of a large number of vehicles to research the average behavior of the vehicles, so as to provide a basis for designing effective traffic control strategies and simulation methods.

A macroscopic traffic flow model by Lighthill and Whitham was first proposed in 1955[1]. Subsequently, Richards[2] also proposed a similar model. These two models are called LWR model and are expressed as follows:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \tag{1}$$

An balanced velocity-density relation is also set so that the equation of the model is closed and the expression is as follows:

$$v = v_{\rho}(\rho) \tag{2}$$

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The main feature of LWR model is that it can describe the formation of traffic shock and the dissipation of traffic congestion. Because its characteristic speed is less than the macro traffic flow speed, LWR model correctly describes the characteristics of various heterogeneity of traffic flow. However, LWR model assumes that the velocity and density phases are always in a fixed curve, i.e. equilibrium state, without considering the dynamic process, so it is unable to describe the physical phase transition and the non-equilibrium traffic flow such as walk stop wave.

In 2000, Aw and Rascle considered the convective derivative of "pressure" instead of its spatial derivative in the acceleration equation[3]. Combining Lighthill and Whitham's hydrodynamic continuity equation in the traffic flow macro model, an Aw-Rascle model with anisotropy was obtained. Later, Rascle improved the model[4], introduced relaxation terms so that the new Aw-Rascle model can avoid gas-like phenomena in the co-directional model. To investigate the need for asymptotic solutions, we introduce the viscous term so that the acceleration equation is:

$$\frac{\partial[v+p(\rho)]}{\partial t}+v\frac{\partial[v+p(\rho)]}{\partial x}=\frac{v_e(\rho)-v}{T}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2} \tag{3}$$

In this paper, the type and stability of equilibrium points of Aw-Rascle model will be further studied by using phase plane analysis theory. The method in this paper extends the range of variables from specific value to infinity by substituting equivalent variables for the original traffic flow model, so as to transform the original model into a new model suitable for phase plane stability analysis, and the traffic flow problem can also be transformed into a system stability analysis problem. The phase plan can also describe a variety of nonlinear phenomena in traffic flow, so as to provide a basis for traffic control decision-making.

The structure of this paper is as follows: in the section 2, the model and its derivation results are discussed; in Section 3, the types and stability of equilibrium points of the model are derived, and the classification and stability of equilibrium points of the model are discussed; in Section 4, numerical simulation is carried out; The fifth section summarizes the full text.

## 2 Model and its derivation

In this paper, the Aw-Rascle model is used to replace the traveling wave equation. In order to visually see the change of traffic flow through the model simulation, the variable substitution is carried out. In addition, the viscous term is added to the model equation, which can make the shock smooth without changing the waveform and contour of the original model. Therefore, the model can be analyzed on the phase plane. The improved Aw-Rascle model used in this paper consists of the following two equations:

Vehicle conservation equation:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \tag{4}$$

And the equations of motion:

$$\frac{\partial v}{\partial t}+\left[v-\rho p^{'}(\rho)\right]\frac{\partial v}{\partial x}=\frac{v_e(\rho)-v}{T}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2} \tag{5}$$

In the equation:

$T$—— relaxation time , $T>0$;
$\mu$—— viscosity term ratio, $\mu>0$;
$Ve[\rho(x,t)]$——speed optimization function;
$p(\rho)$——pressure item, and:

$$p(\rho) = \alpha\left(\frac{\rho}{\rho_m}\right)^\beta \tag{6}$$

This paper assumes that the main road section is an open boundary condition[5], that is:

$$\rho(1,t) = \rho(2,t), \rho(L,t) = \rho(L-1,t), v(1,t) = v(2,t), v(L,t) = v(L-1,t) \tag{7}$$

It is assumed that the model has a sum of traveling wave solutions $\rho(z)$ and $v(z)$, where $z = x - ct$, and the traveling wave velocity c < 0. Using the above results and substituting them into equation (4) (5), we can get:

$$-c\rho_z + q_z = 0 \tag{8}$$

$$-cv_z + (v - c_0)v_z = c(v_e - v) + \mu v_{zz} \tag{9}$$

From formula (8):

$$v_z = \frac{c\rho_z}{\rho} - \frac{q\rho_z}{\rho^2} \tag{10}$$

Substitute formula (10) to formula(9):

$$\frac{\left[v - \rho p'(\rho) - c\right](\rho - q)}{\rho^2}\frac{dy}{dx} = \frac{v_e - v}{T} + \frac{\mu(c\rho - q)}{\rho^3}\left[\frac{d^2\rho}{dz^2} - \frac{2}{\rho}\left(\frac{d\rho}{dz}\right)^2\right] \tag{11}$$

In the formula (10), the integral, namely:

$$-c\rho + q = const = q_* \tag{12}$$

At the same time, namely:

$$q = q_* + c\rho \tag{13}$$

Substitute formula (13) into equation (11):

$$a_1\rho_{zz} + a_2\rho_z + \rho v_e - q = 0 \tag{14}$$

Among them:

$$a_1 = T\left[c - v_e + \rho p'(\rho)\right](c - v_e) \tag{15}$$

$$a_2 = \frac{T\mu(c - v_e)}{\rho} \tag{16}$$

Let $\tilde{y} = \frac{d\rho}{dz}$, the equation (14) can be transformed into a system of first-order ordinary differential equations (or nonlinear dynamic system):

$$\begin{cases} \dfrac{d\rho}{dz} = \tilde{y} \\ \dfrac{dy}{dz} = g(\rho, c)\tilde{y} + f(\rho, c, q_*) \end{cases} \tag{17}$$

Among them:

$$g(\rho, c) = \frac{-\mu}{\rho\left(c - v_e + \rho p'\right)} \tag{18}$$

$$f(\rho, c, q_*) = \frac{q_* + \rho(c - v_e)}{T[c - v_e + \rho p'(\rho)](c - v_e)} \tag{19}$$

The variables are replaced as follows:

$$\eta = \frac{1}{\rho_m - \rho} \tag{20}$$

By substituting it, the following new traffic flow model is obtained:

$$\begin{cases} \dfrac{d\eta}{dz} = y \\ \dfrac{d\eta}{dz} = \left[\dfrac{2y}{\eta} - G(\eta, c)\right] y + F(\eta, c, q_*) \end{cases} \tag{21}$$

Among them:

$$G(\eta, c) = \frac{\mu\eta}{(\rho_m\eta - 1)\left[c - v_e(\eta) - \alpha\beta\left(1 - \frac{1}{\rho_m\eta}\right)^\beta\right]} \tag{22}$$

$$F(\eta, c, q_*) = \frac{q_*\eta^2 + \eta(\rho_m\eta - 1)[c - v_e(\eta)]}{T\left[c - v_e(\eta) - \alpha\beta\left(1 - \frac{1}{\rho_m\eta}\right)^\beta\right][c - v_e(\eta)]} \tag{23}$$

According to the variable substitution, as long as there is traffic congestion, the vehicle density tends to saturated traffic density, and the state variable $\eta$ tends to infinity. Therefore, we can use the phase plan of variable $\eta$ to clearly reflect the corresponding relationship between traffic congestion and system divergence.

After such variable substitution, the new model can correspond traffic congestion with system instability in the phase plan, so as to turn the traffic flow problem into a system stability analysis problem. The variation range of variableρin the original traffic flow model is $0 - 0.2\text{veh/m}$, so the investigation range of density map is very limited. When the system is instable, the maximum value of the variable is 0.2, and after such variable substitution, the variation range of the variable $\eta$ can reach $+\infty$. As long as there is a small fluctuation in traffic flow, there will be great changes in the variable $\eta$. Moreover, as long as there is traffic congestion, the value of the variable $\eta$ tends to infinity, and the new traffic flow model is also convenient to analyze the traffic phenomenon.

## 3 The type and stability of balance points in the model

Let the right-hand term of the equations be zero, it can be got that $\mathrm{y} = 0$ and $\mathrm{F} = 0$, so that the equilibrium point $(\eta_i, 0)$ is determined. The linear system obtained by Taylor expansion of the equation at the equilibrium point is shown in the equation:

$$\begin{cases} \dfrac{d\eta}{dz} = y \\ \dfrac{d\eta}{dz} = G(\eta_i, c)y + F'(\eta_i, c, q_*)(\eta - \eta_i) \end{cases} \tag{24}$$

The Jacobian characteristic equation of linear system is:

$$\lambda^2 - G_i\lambda - F_i' = 0 \tag{25}$$

Among them, $G_i = G(\eta_i, q_*)$, $F_i{}' = F'(\eta_i, c, q_*)$.

According to the qualitative theory of differential equation, the types of equilibrium points of linear system can be determined as follows: (a) When $F_i{}' > 0$, the equilibrium point is the saddle point; (b) When $G_i^2 - 4F_i' > 0$ and $F_i' < 0$, the equilibrium point is the node; (c) When $G_i^2 - 4F_i' < 0$ and $G_i \neq 0$, the equilibrium point is focal point; (d) When $F_i' > 0$ and $G_i = 0$, the equilibrium point is core point. When $z \to \pm\infty$, the stability of the linear systems at the saddle point is all instable; when $G_i < 0$ (or $G_i > 0$ ), the stability at the node or focal point is steady, for $z \to +\infty$ (or $z \to -\infty$).

According to Hartman-Gorban linearization theorem, nonlinear systems and linear systems have the same equilibrium points. For a non-central equilibrium point, the stability at the equilibrium point is consistent between the nonlinear system and the linear system. Given any set of values of traveling wave velocity c and traveling wave parameter q*, the equilibrium point $\eta_i (i = 1,2,\dots)$ of the linear system can be solved. Based on the discussion above, we can determine the types of these equilibrium points and their stability.

**Table 1.** Types of equilibrium points and their stability.

| | $\eta 1$ | $\eta 2$ |
|---|---|---|
| **(c, q*)= (-1.371, 0.64)** | 5.6287<br>$\Delta_i$<0, $G_i < 0$, spiral point<br>Stable for $z \to +\infty$, Instable for $z \to -\infty$ | 7.113<br>F' > 0, saddle point<br>Instable for $z \to \pm\infty$ |
| **(c, q*)= (-6.7, 0.2)** | 5.143<br>$\Delta_i$<0, $G_i < 0$, spiral point<br>Stable for $z \to +\infty$, Instable for $z \to -\infty$ | |

## 4 Numerical simulation

Because the analytical solution of the nonlinear system is not easy to obtain, we select four sets of parameters in the table to simulate the stability at the equilibrium point of the nonlinear system. The phase plane near the equilibrium point is shown in the figure. Where the balance point $(\eta_i ,0)$ is, and $i = 1,2$, $\eta_1 < \eta_2$, as shown by the red point in the figure.

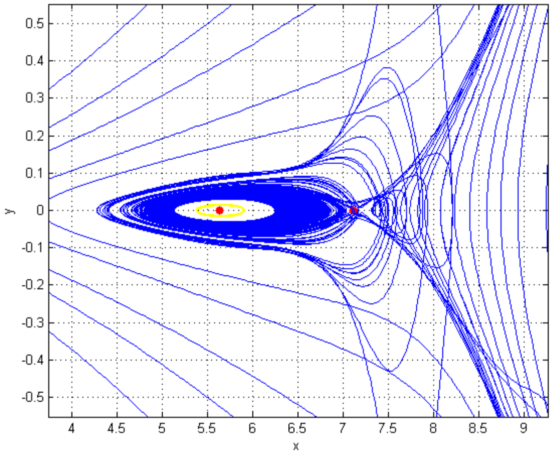


**Fig. 1.** Phase plane trajectory diagram, where traveling wave velocity c=-1.371, traveling wave parameter q*=0.64.

Figure 1 corresponds to the first situation in Table 1. Examine the starting spiral trajectory, when $z \to +\infty$, the orbit trend to the spiral point $(\eta_1 ,0)$, whenz $\to -\infty$, the orbit trend to the outermost circlespiral of the curve which the yellow line outside the spiral point $(\eta_1 ,0)$, far from the spiral point eventually tends to infinity. Thus show, whenz $\to$

$+\infty$ , the system is stable at $(\eta_1, 0)$; when $z \to -\infty$ the system is instable at $(\eta_1, 0)$. When $z \to \pm\infty$, the system is instable at $(\eta_2, 0)$, its nearby track line is far away from the point.

The spiral trajectory approaching system is instable at the equilibrium point $(\eta_2, 0)$. As long as the system becomes instable, the state variable $\eta$ will tend to infinity. Because $\eta$ and $\rho$ are monotonically increasing, the vehicle density will eventually tend to congestion.

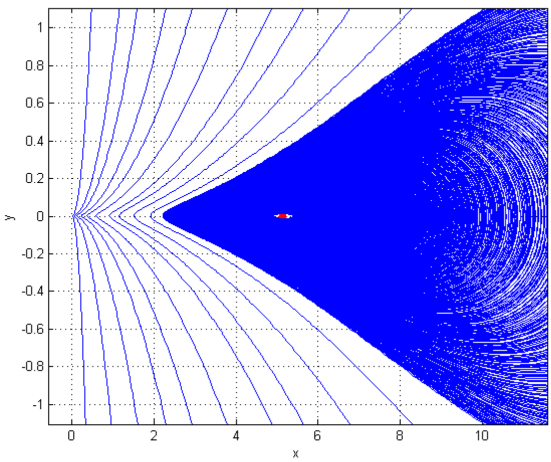


**Fig. 2.** Phase plane trajectory diagram, where traveling wave velocity c=-6.7, traveling wave parameter q*=0.2.

Figure2 corresponds to the second situation in Table 1, whenz $\to \pm\infty$, the system is instable at $(\eta_1, 0)$, its nearby track line is far away from the point, the variable value eventually tends to infinity which indicates that the transportation system is instable and will eventually tend to be congested.

## 5 Conclusion

In this paper, the Aw-Rascle model is transformed by traveling waves and variables, and a new model for analyzing traffic conditions is obtained. The type and stability of the equilibrium solution of the model are also studied. After detailed analysis of the balance point type and stability, our results are verified by data simulation. Finally, the traffic conditions of the improved Aw-Rascle model are further analyzed using the image format of variable substitution.

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