Analysis of equilibrium point based on viscous traffic flow model

Wenhuan Ai[1,*], *Yunfei* Ma[1], and *Dawei* Liu[2]

1College of Computer Science and Engineering, Northwest Normal University, Lanzhou, Gansu, 730070, China
[2]College of Electrical Engineering, Lanzhou Institute of Technology, Lanzhou, Gansu, 730050, China

Abstract. In this paper, the phase plane analysis method is used, which can describe and predict the nonlinear traffic phenomenon on Expressway from the perspective of system global stability. The nonlinear system of the model is obtained by the traveling wave substitution and Taylor expansion of the model, and the equilibrium point of the model is solved by specifying the model parameters. According to the definite theory of differential equation, the model is further analyzed to judge the type and stability of equilibrium point. Finally, numerical verification is carried out according to the simulation diagram. Through numerical verification, the simulation results of the model are consistent with the conclusions of theoretical analysis. It can more clearly describe the change of density or speed with time or road section.

Keywords: Traveling wave solution, Nonlinear traffic phenomena, Traveling wave substitution.

1 Introduction

The ultimate goal of complex nonlinear traffic phenomenon research is to solve and prevent traffic congestion. A large number of researchers have found many such problems in a variety of complex traffic environment, such as traffic hysteresis, lag effect, synchronous flow, stop and go traffic, large-scale mobile congestion, traffic bottleneck, shock wave and sparse wave. Among them, time travel time stop is an important research hotspot in this field. In 1971, Kuner[1] proposed a higher-order continuous model to describe the typical stop and go phenomenon.

Complex nonlinear traffic phenomena have been the focus of research in recent years, in order to alleviate and prevent traffic congestion[2-8]. So far, few people have studied the equilibrium stability of macro traffic flow model. AI et al. Earlier proposed the phase plane analysis method to study the macro traffic flow model. In the bifurcation analysis of the velocity gradient continuum traffic flow model in 2015[9], the traveling wave solution of the new model was obtained by using the phase plane analysis method, and the type and

* Corresponding author: wenhuan618@163.com

stability of the equilibrium point were analyzed and measured. This method can describe and predict the nonlinear traffic phenomena on Expressway from the perspective of system global stability. Through variable substitution, the macro traffic flow model is transformed into a new model suitable for phase plane analysis. The traffic flow problem can also be transformed into the stability analysis of the system. When the traffic flow appears stop and go traffic phenomenon and the traffic flow fluctuation tends to be unstable, there will be curves diverging to infinity on the phase plane, and many curves tend to infinity, while the most stable traffic conditions with small fluctuation are only concentrated in a small area close to the initial value. The phase plan can describe various nonlinear phenomena in traffic flow and provide basis for traffic control decision-making.

The rest of this article is organized as follows. In the second section, we discussed the model and its derivation. In the third section, we deduced the balance point type and stability of the model, and discussed the classification and stability of the model balance point. In the fourth section, a numerical simulation is performed. The fifth part summarizes the full text.

2 Model and its derivation

Next, we take the model proposed by Kerner et al. As an example to analyze the branching phenomenon of traffic flow. The model consists of the following two equations: local vehicle number conservation equation and local vehicle number conservation equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \tag{1}$$

and an equation of motion:

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{C^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=\frac{V_e(\rho)-v}{T}+\mu_0\frac{\partial^2 v}{\partial x^2} \tag{2}$$

Suppose that there are traveling wave solutions $\rho(z)$ and $\mathrm{v}(z)$ in the model, where $\mathrm{z}=x-ct$ is the traveling wave velocity $\mathrm{c}<0$. by using the above results and substituting them into equations (1) and (2), we can get the following results:

$$-c\rho_z+q_z=0 \tag{3}$$

$$-c\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial z}+\frac{C^2(\rho)}{\rho}\frac{\partial \rho}{\partial z}=\frac{1}{T}(\mathrm{V}_e(\rho)-v)+\mu_0\frac{\partial^2 v}{\partial x^2} \tag{4}$$

From formula (3):

$$v_z=\frac{c\rho_z}{\rho}-\frac{q\rho_z}{\rho^2} \tag{5}$$

$$v_{zz}=\frac{c\rho-q}{\rho^2}\rho_{zz} \tag{6}$$

Then, take (5)-(6) into (3) and rewrite (3), we can get:

$$\rho_{zz} + \rho_z(\frac{c}{\mu_0} - \frac{c\rho + q_*}{\rho\mu_0} + \frac{C^2(\rho)\rho}{\mu_0 q_*}) - \frac{\rho(v_e\rho - c\rho - q_*)}{T\mu_0 q_*} = 0 \tag{7}$$

Simplify the second order ordinary differential equation about $\rho(z)$:

$$\rho_{zz} - G(\rho, q_*)\rho_z - F(\rho, c, q_*) = 0 \tag{8}$$

Among them:

$$G(\rho, c, q_*) = -(\frac{c}{\mu_0} - \frac{c\rho + q_*}{\rho\mu_0} + \frac{C^2(\rho)\rho}{\mu_0 q_*}) \tag{9}$$

$$F(\rho, c, q_*) = \frac{\rho(v_e\rho - c\rho - q_*)}{T\mu_0 q_*} \tag{10}$$

Let $y = \frac{d\rho}{dz}$, equation (10) can be transformed into a system of first-order ordinary differential equations:

$$\begin{cases} \frac{d\rho}{dz} = y \\ \frac{dy}{dz} = G(\rho, q_*)y + F(\rho, c, q_*) \end{cases} \tag{11}$$

3 The balance point type and stability of the model

The linear representation of the system can be obtained by Taylor expansion of equation (11):

$$\begin{cases} \rho^{'} = y \\ y^{'} = G(\rho_i, q_*)y + F^{'}(\rho_i, c, q_*)(\rho - \rho_i) \end{cases} \tag{12}$$

Therefore, the Jacobian matrix of the system at the equilibrium point can be obtained as:

$$L = \begin{bmatrix} 0 & 1 \\ F_i^{'} & G_i \end{bmatrix} \tag{13}$$

The corresponding characteristic equation is:

$$\lambda^2 - G_i\lambda - F_i^{'} = 0 \tag{14}$$

Where $G_i(\rho_i, c, q_*) = G(\rho, c, q_*)$ and $\mathrm{F}^{'} = F(\rho, c, q_*)$ get:

$$G_i(\rho_i, c, q_*) = -(\frac{c}{\mu_0} - \frac{c\rho_i + q_*}{\rho_i\mu_0} + \frac{C^2(\rho)\rho_i}{\mu_0 q_*}) \tag{15}$$

$$F` = -\frac{\rho_i}{T\mu_0 q_*}[c - \rho_i V_e'(\rho_i) - V_e(\rho_i)] \quad (16)$$

Since at the equilibrium point $(\rho_i, 0)$, $F = 0$, then $q_* + c\rho_i - \rho_i V_e(\rho_i) = 0$, then F_i' can be written as:

$$F_i' = \frac{q_* + \rho_i^2 V_e'(\rho_i)}{T\mu_0 q_*} \quad (17)$$

From the Hartman-Gorban linearization theorem, we know that the nonlinear system (11) and the linear system (12) have the same equilibrium point. Select the balance velocity function proposed in [10]:

$$V_e[\rho] = v_f\{[1 + \exp(\frac{\rho/\rho_m - 0.25}{0.06})]^{-1} - 3.72\times10^{-6}\}$$

Here, v_f represents the free flow velocity, and ρ_m Represents the maximum density.

The values of the parameters in the model in this chapter are as follows: $v_f = 30\text{m/s}, \rho_m = 0.2$ veh/m, $T = 10\text{s}$, $c_0 = 11\text{m/s}$, $\mu_0 = 550$. From the above discussion and (11)-(12), the type and stability of the equilibrium point can be judged, as shown in Table 1, where the equilibrium point is represented by $\rho_i (i = 1,2,3)$.

Table 1 Types of equilibrium points and their stability when model parameters are given, $\Delta_i = G_i^2 + 4F_i'$, $i = 1,2$.

	ρ_1	ρ_2	ρ_3
	0.0065	0.0938	0.1447
$(c,q) = (-1.371, 0.2)$	$F_i' > 0$,saddle point Unatable for $z \to \pm\infty$	$\Delta_i < 0$, $G_i < 0$, spiral point Stable for $z \to +\infty$ Unstable for $z \to -\infty$	$F_i' > 0$,saddle point Unatable for $z \to \pm\infty$
	0.0223	0.0594	
$(c,q) = (-1.38, 0.64)$	$F_i' > 0$,saddle point Unatable for $z \to \pm\infty$	$\Delta_i < 0$, $G_i < 0$, spiral point Stable for $z \to +\infty$ Unstable for $z \to -\infty$	

4 Numerical simulation

The two sets of parameters in Table 1 are selected to simulate the stability of the nonlinear system (11) at the equilibrium point. The phase plan near the balance point is shown in Figure 1 and Figure 2. The equilibrium point is $(\rho_i, 0)$, and $i = 1,2,3$, $\rho_1 < \rho_2 < \rho_3$. It can be seen from the figure that the balance point type of the system and the stability changes near the balance point are consistent with the theoretical analysis results in Table 3.1.

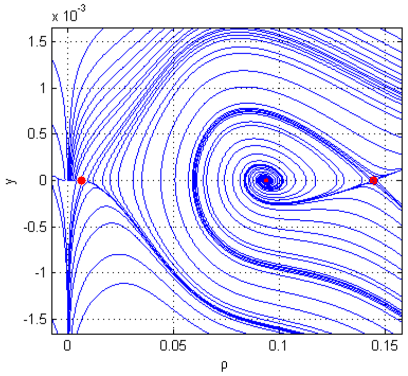

Fig. 1. Phase plane ρ - y trajectory diagram, where traveling wave velocity $c = -1.371$, traveling wave parameter $q_* = 0.2$.

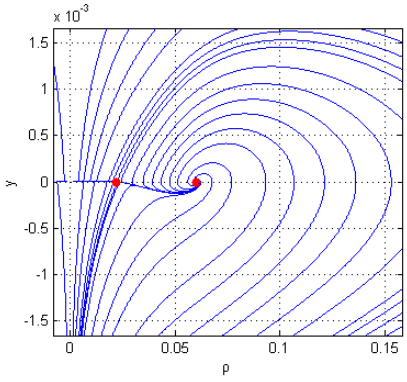

Fig. 2. Phase plane ρ - y trajectory diagram,where traveling wave velocity $c = -1.38$, traveling wave parameter $q_* = 0.64$.

Figure 1 corresponds to the first situation in Table 1. It can be seen from Table 1 that when $z \to \pm\infty$, the system is unstable at the equilibrium point $(\rho_1,0)$, and its nearby trajectories are far away from this point. When $z \to +\infty$, there are several spiral trajectories close to saddle point $(\rho_3,0)$ and tend to focus $(\rho_2,0)$; when $z \to -\infty$, these trajectories are far away from the focus and eventually tend to infinity. It shows that when $z \to +\infty$, the system is stable at $(\rho_2,0)$; when $z \to -\infty$, the system is unstable at $(\rho_2,0)$, the trajectory can be regarded as the system saddle-focus-saddle point solution.

Figure 2 corresponds to the first situation in Table 1. Figure 2 also shows that the system is unstable at the equilibrium point $(\rho_1,0)$. The spiral starting near (0.002,0) tends to focus $z \to +\infty$ at $(\rho_2,0)$, and the system is stable at this point; when $z \to -\infty$, it is far away from focus $(\rho_2,0)$, and the system is unstable.

5 Conclusion

In this paper, the type and stability of equilibrium solutions of Kuhne model are studied by phase plane method. We classify and simulate three types of equilibrium points, and analyze the stability near the equilibrium point. The result can well describe the instability of traffic flow.

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