Analysis of equilibrium point based on nonlinear traffic flow model

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Abstract. In this paper, the traveling wave solution of Kerner konhauser model is obtained by phase plane analysis method, and a new model suitable for stability analysis is obtained by variable transformation. The new model can describe and predict the nonlinear traffic phenomenon on Expressway from the perspective of system global stability. In this paper, the nonlinear system and linear system are obtained respectively by traveling wave substitution and Taylor expansion at the equilibrium point. According to the qualitative theory of differential equations, the equilibrium point type and stability of the linear system are determined. Finally, the simulation results show the consistency between the numerical results and the theoretical analysis.

Keywords: Phase plane diagram, Traveling wave solution, Traveling wave substitution, Nonlinear traffic flow phenomenon, Stability analysis.

1 Introduction

The transportation system is a very complex nonlinear system, which makes a large number of nonlinear phenomena occur in the transportation system. Therefore, researchers need to further analyze the traffic system problem from the perspective of nonlinearity. Researchers found that when the parameters in most traffic flow models change and exceed a certain critical value, the qualitative behavior of the traffic system will also change essentially, which is highly consistent with the sudden changes of various traffic phenomena in the actual traffic, such as free driving state, vehicle walking and stopping state, shock wave, sparse wave The transformation between various traffic states such as road traffic bottleneck, steep drop of traffic capacity, lag effect, cluster effect and synchronous flow. The phenomenon of stopping while walking is a research hotspot. In 1993 Kerner and KonhÄauser[1]proposed a macro high-order traffic flow model to describe the phenomenon of walking and stopping in traffic flow.

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The macro traffic flow model proposed by Kerner and KonhÄauser considers the role of viscosity term on the basis of PW model. PW model is the development of LWR model$^{([2-3])}$.

Since LWR model was proposed, many scholars have further studied it [4] - [5]

The LWR model assumes that the traffic flow always satisfies the equilibrium speed density relationship , so LWR model can not accurately describe the actual traffic flow in non-equilibrium state at most times, and can not simulate nonlinear traffic flow phenomena such as time-to-time stop, ghost traffic congestion and traffic lag, which is its biggest limitation.

In order to overcome the shortcomings of LWR model, many scholars propose to replace the equilibrium velocity density relationship v(x, t) = Ve(ρ (x, t)) with the dynamic equation of average velocity V (x, t), the LWR model is extended to an unbalanced high-order continuous model, $PW^{([6-7])}$ model is a typical one.

Kühne[8]considered the effect of viscosity term on the basis of PW model, that is, added in the acceleration equation$\mu_0 \frac{\varphi^2 v}{\varphi x^2}$, and use c_0^2 replaces the in the Payne model $\frac{\mu}{\rho T}$term.

If the coefficient of the viscous term μ_0 is replaced by the viscosity coefficient inversely proportional to the density to obtain the model proposed by Kerner and $Konh$Ä$auser$[9] in 1993:

$$\begin{cases} \frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \\ \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{V_e(\rho)-v}{T}-\frac{c_0^2}{\rho}\frac{\partial\rho}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2} \end{cases}$$

The model can better explain the traffic phenomena such as time-to-time stop and stability mutation in real traffic. The rest of this paper is organized as follows. In the second part, we discuss the model and its derivation. In the third part, the types and stability of equilibrium points of the model are derived, and the classification and stability of equilibrium points of the model are discussed. The fourth part carries out numerical simulation. The fifth part is the summary of the full text.

2 Model and its derivation

Taking the density gradient continuous traffic flow model proposed by Kerner and KonhÄauser in 1993 as an example, the branching phenomenon of traffic flow is analyzed. The model expression is composed of the following two equations, an equation with local vehicle number conservation

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 \tag{1}$$

And an equation of motion:

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{V_e(\rho)-v}{T}-\frac{c_0^2}{\rho}\frac{\partial\rho}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2} \tag{2}$$

It is assumed that the model has traveling wave solutions ρ (z) and v (z) where z =x -ct and traveling wave velocity c < 0. Using the above results and substituting them into equations (1) and (2), we can get

$$-c\rho_z+q_z=0 \tag{3}$$

$$-c\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial z}=\frac{1}{T}(v_e(\rho)-v)-\frac{C_0^2}{\rho}\frac{\partial \rho}{\partial z}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2} \tag{4}$$

By deriving both ends of equation PV = q at the same time, it is obtained that:

$$q_z=\rho_z v+v\rho_z \tag{5}$$

From (3) and (5):

$$v_z=\frac{c\rho_z}{\rho}-\frac{q\rho_z}{\rho^2} \tag{6}$$

(3) an integral can be obtained for Z.

$$-c\rho+q=const=q^* \tag{7}$$

Bring (6), (7) into (4):

$$\rho_{zz}+\frac{\rho}{u}\left(\frac{c_0^2\rho}{q^*}-\frac{q^*}{\rho}\right)\rho_z-\frac{p^2[q^*+c\rho-\rho v_e(\rho)]}{uTq^*}=0 \tag{8}$$

Simplify the second order ordinary differential equation about $\rho(z)$:

$$\rho_{zz}-g(\rho,q^*)\rho_z-f(\rho,c,q^*)=0 \tag{9}$$

Among them:

$$g(\rho,q^*)=\frac{\rho}{u}\left(\frac{q^*}{\rho}-\frac{c_0^2\rho}{q^*}\right) \tag{10}$$

$$f(\rho,c,q^*)=-\frac{\rho^2}{q^*uT}[q^*+c\rho-\rho v_e(\rho)] \tag{11}$$

Let y=$\frac{dp}{dz}$, equation (9) can be transformed into a system of first-order ordinary differential equations

$$\begin{cases}\dfrac{d\rho}{dz}=\bar{y}\\ \dfrac{d\bar{y}}{dz}=g(\rho,q^*)\bar{y}+f(\rho,c,q^*)\end{cases} \tag{12}$$

Substitute variables as follows:$\theta=\frac{1}{\rho_m-\rho}$
By substituting them into (11), the following new traffic flow models can be obtained

$$\begin{cases}\dfrac{\mathrm{d}\theta}{\mathrm{d}z}=y\\ \dfrac{\mathrm{d}y}{\mathrm{d}z}=\left[\dfrac{2y}{\theta}+G(\theta,q^*)\right]y+F(\theta,c,q^*)\end{cases} \tag{13}$$

Among

$$G(\theta,q^*)=\frac{p_m\theta-1}{u\theta}\left\{\frac{q^*\theta}{p_m\theta-1}-\frac{c_0^2(\rho_m\theta-1)}{q^*\theta}\right\} \tag{14}$$

$$F(\theta,c,q^*)=\frac{-(1-\rho_m\theta)^2}{uT\theta q^*}[\theta q^*+(1-\theta\rho_m)(v_e(\theta)-c)] \tag{15}$$

3 The balance point type and stability of the model

For the analysis of the equilibrium point, it is mainly determined from the equation group formed by equation (13). Let the right end of the equation of equation (13) be 0, it can be seen that $y=0$ and $F=0$, thus the equilibrium point coordinates $(\rho_i, 0)$, Then Taylor expansion of equation (13) according to the coordinates of the equilibrium point, the linear representation of the system can be obtained:

$$\begin{cases} \theta' = y \\ y' = G(\theta_i, q^*)y + F'(\theta_{i,}c, q^*)(\theta - \theta_i) \end{cases} \tag{16}$$

Therefore, the Jacobian matrix of the system at the equilibrium point can be obtained as:

$$L = \begin{bmatrix} 0 & 1 \\ F_i' & G_i \end{bmatrix} \tag{17}$$

The corresponding characteristic equation is:

$$\lambda^2 - G_i\lambda - F_i' = 0 \tag{18}$$

where $G_i(\rho_i, c, q_*) = G(\rho, c, q_*)$, $F^` = F(\rho, c, q_*)$.

Since at the equilibrium point $F=0$, then $\theta_i q^* + (1 - \rho_m\theta_i)(v_e(\theta_i) - c) = 0$,then we have

$$G(\theta_i, c, q^*) = \frac{u\theta_i}{p_m\theta_i - 1}\left[\frac{q^*\theta_i}{\rho_m\theta_i - 1} - \frac{c_0^2(\rho_m\theta_i - 1)}{q^*\theta_i}\right] \tag{19}$$

$$F' = \frac{-(1-\rho_m\theta_i)^2}{uq^*\theta_i T}\left[q^* - \frac{\rho_m\theta_i q^*}{1-\rho_m\theta_i} + (1+\rho_m\theta_i)v_e'(\theta_i)\right] \tag{20}$$

From the Hartman-Gorban linearization theorem, we know that the nonlinear system (13) and the linear system (16) have the same equilibrium point. Select the balance velocity function proposed in [9]:

$$V_e[\rho] = v_f\left\{\left[1 + exp(\frac{\frac{\rho}{\rho m} - 0.25}{0.06})\right]^{-1} - 3.72 \times 10^{-6}\right\} \tag{21}$$

From 21:

$$V_e(\theta) = v_f\left\{\left[1 + exp(12.5 - \frac{1}{0.06\rho_m\theta})\right]^{-1} - 3.72 \times 10^{-6}\right\} \tag{22}$$

Here, v_f represents the free flow velocity, and ρ_m represents the maximum or congestion density.

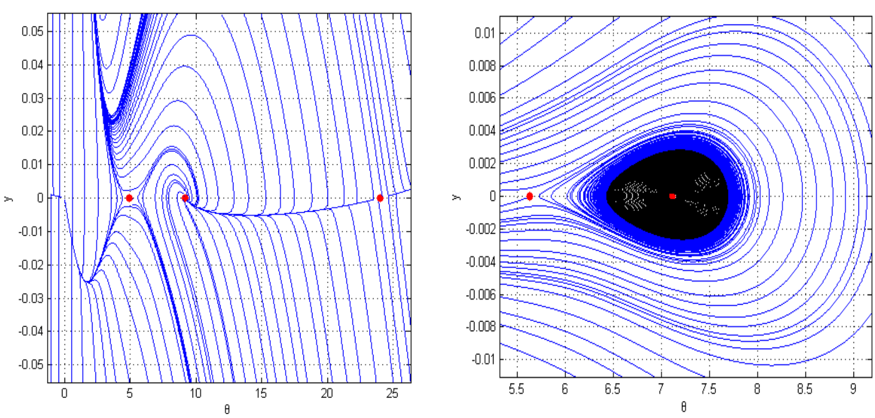
The values of the parameters in the model in this chapter are as follows: v_f = 30m/s, ρ_m = 0.2 veh/m, T = 10s, c_0 = 11m/s, μ_0 = 550. When ρ is equal to 0, this is a trivial balance point and has no practical meaning, so this article only needs to discuss other balance points. From the above discussion and (19)-(20), the type and stability of the equilibrium point can be judged, as shown in Table3. 1, where the equilibrium point is represented by θ_i(i = 1, 2,3).

Table 1. Types of equilibrium points and their stability when model parameters are given, $\Delta_i = G_i^2 + 4F_i^{'}, i = 1,2$.

	θ_1	θ_2	θ_3
	5.1696	9.1994	23.9752
(c,q^*)=(-1.26,0.2)	$F_i^{'} > 0$,saddle point Unatable for $z \to \pm\infty$	$\Delta_i < 0$, $G_i < 0$, spiral point Stable for $z \to +\infty$ Unstable for $z \to -\infty$	$F_i^{'} > 0$,saddle point Unatable for $z \to \pm\infty$
	5.6287	7.113	
(c,q^*)=(-1.371,0.64)	$F_i^{'} > 0$,saddle point Unatable for $z \to \pm\infty$	$\Delta_i < 0$, $G_i < 0$, spiral point Stable for $z \to +\infty$ Unstable for $z \to -\infty$	

4 Numerical simulation

The two sets of parameters in Table 1 are selected to simulate the stability of the nonlinear system (13) at the equilibrium point. The phase plan near the balance point is shown in Figure 1. The equilibrium point is $(\theta_i, 0)$, and $i = 1,2,3$, $\theta_1 < \theta_2 < \theta_3$. It can be seen from the figure that the balance point type of the system and the stability changes near the balance point are consistent with the theoretical analysis results in Table 1.



a b

Fig. 1. Phase plane $\theta - y$ trajectory diagram, (a).where traveling wave velocity c=-1.26,traveling wave parameter $q_* = 0.2$. (b).where traveling wave velocity c=-1.371,traveling wave parameter q*=0.64.

Figure a corresponds to the first group of data in Table 1. when $z \to \pm\infty$, the system is unstable at the equilibrium point(θ_1 ,0) and(θ_3 ,0), and its nearby trajectories are far away from this point. When $z \to +\infty$, there are several spiral trajectories close to saddle point (θ_1 ,0) and tend to focus(θ_2 ,0); due to simultaneous influence of nearby saddle point (θ_3 ,0) and Therefore, the spiral state of trajectory tending to focus is not obvious; The rotation state is not obvious; When $z \to -\infty$, these trajectories are far away from the focus and eventually tend to infinity. These trajectories are the saddle point focus solution of the system. It shows that when $z \to +\infty$, the system is stable at (θ_2 ,0); When $z \to -\infty$,The system is unstable at (θ_2 ,0). As long as the system becomes unstable, the state variable θ will tend to infinity ,due to θ and ρ In a monotonic increasing relationship, the vehicle density will eventually tend to the state of congestion.

Figure b corresponds to the second group of data in Table 1, and also shows that the system is unstable at the equilibrium point(θ_1 ,0) when $z \to \pm\infty$. he spiral trajectory starting from (6.8,0) tends to focus (θ_2 ,0) when $z \to +\infty$, when $z \to -\infty$, stay away from the Focus and finally form equal amplitude oscillation. It is further found that the spiral trajectory starting from (6.4,0) tends to change when $z \to -\infty$ Near the outermost circle of the curve shown by the black line; When $z \to +\infty$, it tends to infinity. So there is a gap between the black line and the blue line have a limit cycle

In order to facilitate the simulation, the space spacing is 160m and the time interval is 1s. The values of other parameters in the model are as follows: T=10s, c_0=11m/s,u=550, $v_f = 30m/s, \rho_m$=0.2veh/m.

Select the Hopf branch point ρ_0 = 0.058789veh/m as the initial uniform density value, the applied amplitude is for local small disturbance of $\Delta \rho_0$= 0.01 veh/ m, draw the density space-time diagram of the system, as shown in Figure 2: It can be seen from Fig. 4-2 that the amplitude of the initial small disturbance increases with time and evolves into constant amplitude periodic oscillation, which is the characteristic of the limit cycle solution. It also reflects the traffic phenomenon of oscillation and congestion, and stops the wave immediately. It also further verifies the correctness of the theoretical analysis.

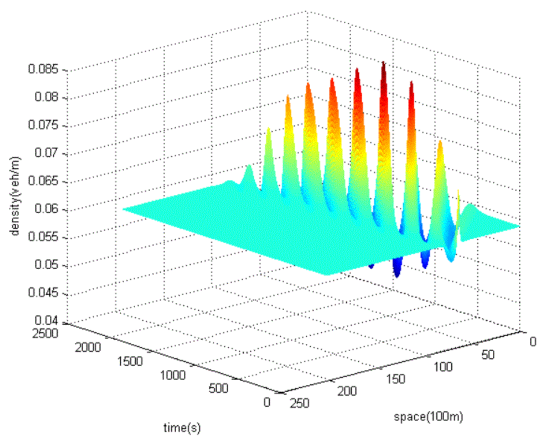


Fig. 2. Density space-time diagram with Hopf branch as initial value.

5 Conclusion

In this paper, the traveling wave solution of Kerner konhauser model is obtained by phase plane analysis method, and then the equilibrium point type and stability of the system are analyzed by the new model after substitution. Finally, by drawing the density space-time map and phase plan of the system, the traffic phenomena such as time-to-time stop and stability mutation on the expressway can be better explained

The authors would like to thank the anonymous referees and the editor for their valuable opinions. This work is partially supported by the National Natural Science Foundation of China under the Grant No. 61863032 and the Natural Science Foundation of Gansu Province of China under the Grant No. 20JR5RA533 and the China Postdoctoral Science Foundation Funded Project (Project No.: 2018M633653XB) and the "Qizhi" Personnel Training Support Project of Lanzhou Institute of Technology (2018QZ-11) and Gansu Province Educational Research Project (Grant No. 2021A-166)

Reference

1. B.S. Kerner, P. Konh¨auser. Cluster effect in initially homogeneous traffic flow [J]. Physical Review E 1993, 48: R2335-R2338.
2. M.J. Lighthill, G.B. Whitham. On kinematic waves: II. A theory of traffic flow on long crowed roads [C]. In Proceedings of the Royal Society, London 1955, 229: 317-345.
3. P.I. Richards. Shocks waves on the highway [J]. Operations research 1956, 4: 42-51.
4. J.H. Bick, G.F. Newell. A continuum model for two-directional traffic flow [J]. Quarterly of Applied Mathematics 1960, 18: 191-204.
5. P.G. Michalopoulos, G. Stephanopoulos, V.B. Pisharody. Modeling of traffic flow at signalized links [J]. Transportation Science 1980, 14: 9-41.
6. H.J. Payne. Models of freeway traffic and control [C]. La Jolla, Calif: Mathematical Models of Public Systems, Simulation Councils, 1971, 1: 51-61.
7. G.B. Whitham. Linear and nonlinear waves [M]. New York: Wiley-Interscience, 1974.
8. R.D. KÄuhne. Macroscopic freeway model for dense traffic-stop-start waves and incident detection [C]. Delft: Proceedings of the Ninth International Symposium on Transportation and Traffic Theory 1984: 20-42.
9. B.S. Kerner, P. KonhÄauser. Cluster effect in initially homogeneous traffic flow [J]. Physical Review E, 1993, 48: 2335-2338