Stability analysis of a viscous continuous traffic flow model

Wenhuan Ai*, *Na* Li, and *Ruihong* Tian

College of Computer Science and Engineering, Northwest Normal University, Lanzhou, Gansu, 730070, China

Abstract. This paper studies the stability of a speed gradient continuous traffic flow model, which is proposed by Ge et al and based on TVDM. The nonlinear and linear systems of traveling wave solutions of the model equation are derived by traveling wave substitution. And the types of equilibrium points and it's stability are analyzed theoretically. Finally, the phase plane diagram is obtained through simulation, and the global distribution structure of the trajectories is analyzed. The results show that the numerical results are consistent with the theoretical analysis, so some nonlinear traffic phenomena can be analyzed and predicted from the perspective of global stability.

Keywords: Macroscopic model, Traffic flow, Equilibrium point, Stability analysis.

1 Introduction

In order to alleviate and prevent traffic congestion fundamentally, people have studied the various nonlinear phenomena of traffic flow for decades. However, it is not enough to improve traffic congestion only by increasing traffic infrastructure. We should study and analyze the traffic flow theory and deeply investigate the internal mechanism of traffic phenomenon, which can help us dredge and control the traffic congestion phenomenon fundamentally. In addition to evaluate all kinds of traffic phenomena well by observing experiments, researchers have also reproduced and explained the traffic phenomena through a large number of model equations. Compared with car-following model, macro traffic flow model can better describe the collective behavior of vehicles in traffic flow, so as to alleviate and prevent traffic congestion to a greater extent.

As early as 1955, Lighthill and Whitham[1] first proposed a macroscopic traffic flow model. In 1956, Richards[1] proposed a similar model. These two models are called LWR models. In 1971, Payne[2] proposed the first macroscopic high-order continuous model, which allows the average velocity of the vehicles to deviate from the equilibrium velocity. Later, Kühne[4] added a viscous term in the acceleration equation, it can smooth the discontinuity in the model. But in 1995, Daganzo[5] found that there is always a characteristic velocity greater than the macro flow velocity in these models, which aganist a

* Corresponding author: wenhuan618@163.com

basic principle of traffic flow. In order to overcome this problem, researchers proposed some anisotropic macroscopic traffic flow models. In 2000, Aw[6] first proposed the anisotropic speed gradient model. After that, some similar macro anisotropic models had been proposed[7-10].

At present, the research on macro traffic flow mainly focus on analyzing the stability by using characteristic analysis method, deriving the solution equations of different regions[8,9,10]. There are few studies on the equilibrium point and its stability. Therefore, this paper studies and analyzes the macro model from this point of view. Firstly, we can study the type and stability of equilibrium points in macroscopic traffic flow model, then we can study the most common traffic flow state instability phenomenon from the equilibrium point, so it is particularly necessary to study the equilibrium point and stability of traffic flow.

The rest of this paper is organized as follows. In Section 2, the model is derived. The properties of the equilibrium points are analyzed in Section 3. The numerical simulation is carried out in Section 4, and finally summarized.

2 Model and it's derivation

In this paper, we study and analyze a new speed gradient continuous traffic flow model, which is based on TVDM and proposed by Ge[10] et al. The expression is as follows:

$$\begin{cases} \partial\rho/\partial t + \partial(\rho v)/\partial x = 0 \\ \partial v/\partial t + v\partial v/\partial x = [V_e(\rho) - v]/T + c(\rho)\partial v/\partial x + \mu(\rho)\partial^2 v/\partial x^2 \end{cases} \tag{1}$$

where ρ is the average density, v is the average velocity, T is the relaxation time, $V_e(\rho)$ is the equilibrium velocity, $c(\rho) = \Delta/\tau$ is the disturbance propagation velocity, $\mu(\rho) = (1-p)\tau c^2(\rho)$ is viscosity coefficient, p is the weight coefficient. τ is the time needed for the backward propagated disturbance to travel a distance of Δ.

Supposing that there are traveling wave solutions $\rho(z)$ and $v(z)$ in the model, where $z = x - ct$ and the traveling wave velocity $c < 0$. Substituting them into the Eq. (1), we can get the following results.

$$-c\rho_z + \rho v_z + v\rho_z = 0 \tag{2}$$

$$-cv_z + (v - c_0)v_z = [V_e(\rho) - v]/T + \mu_0 v_{zz} \tag{3}$$

The two sides of Eq. (2) are integral to z, we have

$$-c\rho + q = const = q_*$$

$$q = q_* + c\rho \tag{4}$$

From Eq. (4), we can get

$$v = \frac{q}{\rho} = \frac{q_*}{\rho} + c \tag{5}$$

Substitute Eq. (4) into Eq. (3)

$$-c(-\frac{q_*}{\rho^2})\rho_z+(\frac{q_*}{\rho}+c-c_0)(-\frac{q_*}{\rho^2})\rho_z=\frac{1}{T}(V_e-\frac{q_*}{\rho}-c)+\mu_0(-\frac{q_*}{\rho^2})\rho_{zz} \tag{6}$$

The second order ordinary differential equation about $\rho(z)$ can be obtained by simplifying.

$$\rho_{zz}-g(\rho,q_*)\rho_z-f(\rho,c,q_*)=0 \tag{7}$$

where

$$g(\rho,q_*)=\frac{1}{\mu_0}(\frac{q_*}{\rho}-c_0) \tag{8}$$

$$f(\rho,c,q_*)=\frac{\rho}{T\mu_0 q_*}[\rho V_e(\rho)-q_*-c\rho] \tag{9}$$

Let $y=d\rho/dz$, Eq. (7) can be transformed into a system of first order ordinary differential equations (or called nonlinear dynamic system) with respect to y.

$$\begin{cases} d\rho/dz=y \\ dy/dz=g(\rho,q_*)y+f(\rho,c,q_*) \end{cases} \tag{10}$$

3 Equilibrium and stability

If the right term of the system (10) is zero, we can obtain $y=0$ and $f=0$. So the equilibrium point $(\rho_i,0)$ can be determined, ρ_i for short. The right term of the second equation in the system (10) can also be linearized through the Taylor expansion at the equilibrium point, and the higher order term is ignored, then the linearized system of Eq. (10) can be derived as follows:

$$\begin{cases} d\rho/dz=y \\ dy/dz=g(\rho_i,q_*)y+f'(\rho_i,c,q_*)(\rho-\rho_i) \end{cases} \tag{11}$$

We obtain the Jacobian matrix of the system (11) at the equilibrium points as

$$L=\begin{bmatrix} 0 & 1 \\ f_i' & g_i \end{bmatrix} \tag{12}$$

and the Jacobian characteristic equation as

$$\lambda^2-g_i\lambda-f_i'=0 \tag{13}$$

where $g_i=g(\rho_i,q_*)$, $f_i'=f'(\rho_i,c,q_*)$.

Since $f=0$ at the equilibrium point, we have $\rho_i V_e(\rho_i)-q_*-c\rho_i=0$, then we can get

$$g_i=\frac{1}{\mu_0}(\frac{q_*}{\rho_i}-c_0) \tag{14}$$

$$f_i^{'} = \frac{q_* + \rho_i^2 V_e^{'}(\rho_i)}{T\mu_0 q_*} \tag{15}$$

According to the qualitative theory of differential equation, the types of equilibrium point of linear system (11) can be determined, as shown in table 1.

Table 1. Types of equilibrium points for linear dynamical systems (11).

$f_i^{'}$	$g_i^2+4f_i^{'}$	g_i	Types of the equilibrium points
+			Saddle point
-	+		Nodal point
-	-	$\neq 0$	Spiral point
-	-	0	Central point

The linear system(11) is unstable at the saddle point when $z \to \pm\infty$; and stable at the nodal or spiral point for $z \to +\infty$ (or $z \to -\infty$) when $g_i < 0$ (or $g_i > 0$). According to the Hartman-Gorban linearization theorem[11], when the eigenvalues of nonlinear system have non-zero real parts, that is, the equilibrium point is a non central point, the stability of nonlinear system and linear system at equilibrium point is consistent. It is difficult to appear a central point in the traffic flow model, so we discussed it here.

4 Numerical simulation

The equilibrium point of linear system (11) can be solved for any given set of c and q_*. Then according to table 1, the type of equilibrium point and its stability can be determined and are shown in table 2. Equilibrium velocity function proposed in Ref. [12] is selected for numerical simulation.

$$V_e[\rho] = v_f \left\{ \left[1 + \exp\left(\frac{\rho/\rho_m - 0.25}{0.06} \right) \right]^{-1} - 3.72\times10^{-6} \right\} \tag{16}$$

Other parameters in the model are as follows:

$$v_f = 30m/s,\quad \rho_m = 0.2veh/m,\quad T = 10s,\quad p = 0.86,\quad c(\rho) = 11m/s,\quad \mu(\rho) = 154$$

Table 2. Types and stabilities of some equilibrium points of linear system (11) with the given model parameters, where $\Delta_i = g_i^2 + 4f_i^{'}, i = 1,2,3$.

(c,q_*)	ρ_1	ρ_2	ρ_3
(-1.371,0.36)	0.0119 $f_i^{'}>0$,saddle point Unatable for $z\to\pm\infty$	0.0744 $\Delta_i<0$, $g_i<0$,spiralpoint Stable for $z\to+\infty$ Unstable for $z\to-\infty$	0.2626 $f_i^{'}>0$,saddle point Unatable for $z\to\pm\infty$
(-1.38,0.80)	0.0301 $f_i^{'}>0$,saddle point Unatable for $z\to\pm\infty$	0.0512 $\Delta_i<0$, $g_i<0$,spiral point Stable for $z\to+\infty$ Unstable for $z\to-\infty$	

Using the two sets of parameters in table 2 to simulate the system (10) respectively. We can obtain the phase plane diagrams near the equilibrium point as shown in figure 1. The corresponding equilibrium points are marked in the phase plane diagrams.

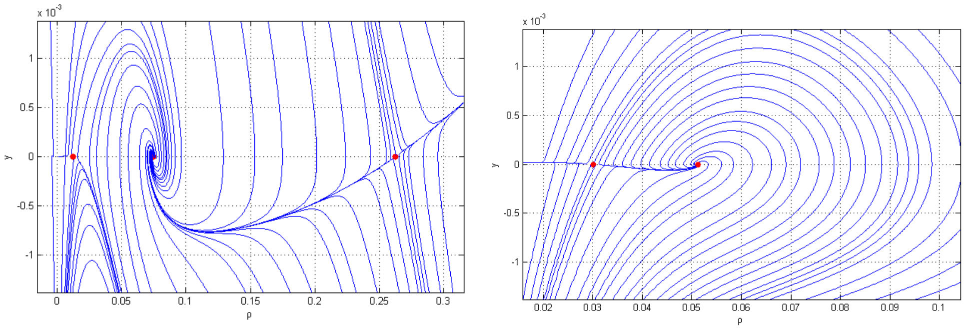

Fig. 1. Trajectories in $\rho - y$ phase plane with the parameters: (a) $c=-1.371$ and $q_*=0.36$; (b) $c=-1.38$ and $q_*=0.80$.

It can be seen from figure 1.(a) that $(\rho_1,0)$ and $(\rho_3,0)$ are saddle points, when $z\to\pm\infty$, the trajectory is far away from this point, and the system is unstable at this point. $(\rho_2,0)$ is spiral point, when $z\to+\infty$, there are several spiral trajectories close to the saddle points tend to focus on $(\rho_2,0)$, indicating that the system is stable; when $z\to-\infty$, these trajectories are far away from spiral point and finally tend to infinity, indicating that the system is unstable at that point, which can be regarded as the saddle point-spiral point-saddle point solution of the system. In figure 1.(b), $(\rho_1,0)$ is saddle point, when $z\to\pm\infty$, the system is unstable; $(\rho_2,0)$ is the spiral point, the trajectories starting from (0.03,0) tends to focus on the spiral point for $z\to+\infty$, indicating that the system is stable; when $z\to-\infty$, these trajectories are far away from the spiral point, the system is unstable. Finally, the results show that the numerical results are consistent with theoretical analysis. The trajectories show the global structure of the interaction of multiple equilibrium points, reflecting the influence of different types of equilibrium points on the trajectories around them.

5 Conclusion

In this paper, we studied a new speed gradient continuous traffic flow model based on TVDM, and discussed the types of equilibrium points and their stability states. Numerical simulation on the phase plane shows that the numerical results are consistent with the theoretical analysis. The results is helpful to describe and study nonlinear traffic phenomena.

The authors would like to thank the anonymous referees and the editor for their valuable opinions. This work is partially supported by the National Natural Science Foundation of China under the Grant No. 61863032 and the Natural Science Foundation of Gansu Province of China under the Grant No. 20JR5RA533 and the China Postdoctoral Science Foundation Funded Project (Project No.: 2018M633653XB) and the “Qizhi” Personnel Training Support Project of Lanzhou Institute of Technology (2018QZ-11) and Gansu Province Educational Research Project (Grant No. 2021A-166).

References

1. Lighthill M J, Whitham G B. On Kinematic Waves. II. A Theory of Traffic Flow on Long Crowded Roads[J]. Proceedings of The Royal Society A Mathematical Physical and Engineering Sciences, 1955, 229(1178):317-345.

2. Richards, Paul I.Shock Waves on the Highway[J]. Operations Research, 1956, 4(1):42-51.
3. Payne H J. Models of Freeway Traffic and Control[J]. Mathematical Models of Public Systems Simulation Council, 1971, 1:51-61.
4. Kühne R D. Macroscopic Freeway Model for Dense Traffic: Stop-Start Waves and Incident Detection[C]. International Symposium on Transportation and Traffic Theory. 1984.
5. Daganzo C F. Requiem for second-order fluid approximations of traffic flow[J]. Transportation Research Part B: Methodological, 1995, 29, 277-286.
6. Aw, A, Rascle, et al. Resurrection of "Second Order" Models of Traffic Flow[J]. Siam Journal on Applied Mathematics, 1999.
7. Zhang H M. A theory of nonequilibrium traffic flow[J]. Transportation Research Part B: Methodological, 1998, 32(7):485-498.
8. Jiang R, Wu Q S, Zhu Z J. A new continuum model for traffic flow and numerical tests[J]. Transportation Research Part B, 2002, 36(5):405-419.
9. Lai L L, Cheng R J, Li Z P, et al. The KdV-Burgers equation in a modified speed gradient continuum model[J]. Chinese Physics B, 2013, 22(6):060511.
10. Ge H X, Lo S M. The KdV-Burgers equation in speed gradient viscous continuum model[J]. Physica A Statistical Mechanics & Its Applications, 2012, 391(4):1652-1656.
11. Ai W H, Shi Z K, Liu D W. Bifurcation analysis of a speed gradient continuum traffic flow model[J]. Physica A Statistical Mechanics & Its Applications, 2015, 437:418-429.
12. Kerner B S, Konhäuser P. Cluster effect in initially homogeneous traffic flow[J]. Physical Review E, 1993, 48(4):R2335-R2338.