Existence of Gauss John ellipsoid operator problem

Li'ao Yu, and *Du* Zou*

College of Science, Wuhan University of Science and Technology, Wuhan, Hubei, 430065, China

Abstract. There is a common example in convex geometry and Banach space geometry: the unique ellipsoid with the largest volume associated with each symmetric convex body K is called John ellipsoid. In this paper, we will give some results of John ellipsoid operator problem with Gaussian measure, mainly the continuity of Gaussian measure and the existence of ellipsoid operator.

Keywords: Convex geometry, Symmetric convex body, John ellipsoid, Gaussian measure.

1 Introduction

1.1 John ellipsoid

John ellipsoid indicates that each convex body K is associated with a unique ellipsoid with the largest volume. Given a convex body K in R^n, Find the only ellipsoid among all ellipsoids E to solve the following maximization problem:

$$\sup_{E}|E| \quad E\subseteq K \tag{1}$$

John ellipsoid [1][2] is the most typical affine associative ellipsoid operator in convex geometry,Its generalization in functional analysis is Lewis ellipsoid operator, It is a tool in Banach space local theory (asymptotically convex geometric analysis) and PDEs. Its application has reached the fields of optimization theory, cybernetics, computer image recognition, information theory and so on. The appearance of John ellipsoid creates an important branch of extreme value problem in convex geometry. On the basis of John ellipsoid, many mathematicians have made unremitting efforts to expand the important concept of John ellipsoid to a new stage. For example: Petty ellipsoid; Lp John ellipsoid [3] and Orlicz John ellipsoid [4]. Lp John ellipsoid combines John ellipsoid with Minkowski theory, which successfully classifies ellipsoids. When $p=1$, it is the Petty ellipsoid

* Corresponding author: wangwenbo@wust.edu.cn

mentioned above; When $p = 2$, the result studied by LYZ is called LYZ ellipsoid; When $p = \infty$, it is the classical John ellipsoid.

1.2 Introduction to Gauss volume

Definition of standard Gaussian volume

$$\gamma_n(E) = c_n \int_E e^{-\|x\|/2} dx \tag{2}$$

Where E represents the measurable set on R^n, $\|x\|$ represents Euclidean norm on n-dimensional Euclidean space, dx represents the volume measure, and

$$c_n = (2\pi)^{-n/2} \tag{3}$$

For $n \in N$ and $r \in R$, we can define

$$\gamma_n(sE)^{1/n} \geq s\gamma_n(E)^{1/n} \text{ 若 } 0 \leq s \leq 1 \tag{4}$$

$$\gamma_n(sE)^{1/n} \leq s\gamma_n(E)^{1/n} \text{ 若 } s \geq 1 \tag{5}$$

The equality holds if and only if $s = 1$ or $\gamma_n(E) = 0$

If C is a Borel star set, it can be obtained from polar coordinates

$$\gamma_n(E) = c_n \int_E e^{-\|x\|/2} dx = c_n \int_{S^{n-1}} \int_0^{\rho_C(u)} e^{-r^2/2} r^{n-1} dr du = c_n \int_{S^{n-1}} \phi_n(\rho_C(u))^n du \tag{6}$$

Where $\rho_C(u)$ represents the radial function, and the expression is as follows:

$$\rho_C(u) = \sup\{c \in R : cu \in C\} \tag{7}$$

The specific definition of ϕ_n is as follows:

$$\phi_n(a) = \left(\int_0^a e^{-t^2/2} t^{n-1} dt \right)^{1/n} \tag{8}$$

In recent years, convex geometry is in the process of rapid development, and the theory has gradually integrated and connected with other mathematical branches. As we all know, normal distribution is in the "center" of many probability distributions. Normal distribution is also called Gauss measure or Gauss volume in geometry. Gauss volume, as a geometric functional on convex body class, is associated with Boroczky-Lutwak-Yang-Zhang conjecture proposed in 2012. More importantly, combined with Minkowski theory, Brunn Minkowski inequality under Gaussian measure is obtained [5] [6] [7], Because of the significance of Brunn Minkowski inequality in convex geometry, more and more classical works [8] and excellent results [9] [10] [11] have been published in the past few years. The Brunn Minkowski inequality has also been extended, and various forms of Brunn Minkowski inequality have appeared. Brunn Minkowski inequality under Gaussian measure is one of the typical representatives.

For each symmetric convex set K and L in R^n, and for arbitrary $\lambda \in (0,1)$ The following inequality holds:

$$\gamma_n\left(\lambda K+(1-\lambda)L\right)^{1/n} \geq \lambda\gamma_n(K)^{1/n}+(1-\lambda)\gamma_n(L)^{1/n} \tag{9}$$

The conditions for the establishment of the equality is $K = L$

Another important inequality under Gaussian measure is ehrhard's inequality [12],It shows each Borel measurable set A and B, and for arbitrary $\lambda \in (0,1)$

$$\Phi^{-1}\left(\gamma_n\left(\lambda A+(1-\lambda)B\right)\right) \geq \lambda\Phi^{-1}\left(\gamma_n(A)\right)+(1-\lambda)\Phi^{-1}\left(\gamma_n(B)\right) \tag{10}$$

Where Φ^{-1} represents the inverse of the Gaussian distribution function, The Gaussian distribution function is expressed as follows:

$$\Phi(x)=\gamma_1\left((-\infty,x)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt \tag{11}$$

The emergence of ehrhard's inequality obtains many subtle probability and geometric properties of Gaussian distribution. On this basis, Gardner and zvavitch study the Gaussian inequality in dual Brunn Minkowski theory [13] in detail, and give the specific dual Gaussian Brunn Minkowski inequality.

Through the introduction of John ellipsoid and Gauss measure, we will have such an idea: whether there is also a unique associated ellipsoid operator under Gaussian measure. Next, we will give some important tools in the existence of Gauss John ellipsoid operator problem

Theorem 1 (Blaschke selection theorem). Every bounded sequence in metric space (K^n, δ_H) has a convergent subsequence.

Proof: Let $\{C_j\}_j \subset K^n$ be a bounded sequence. Therefore, there is a n dimensional cube with side length d, which contains all the members in the sequence. Do the following for E: In step i, Decompose cube E into congruent closed "small" cubes with internal disjoint, The sides of these small cubes are $d/2^i$. Write the class composed of these "small" cubes as B_i.

For each i, For each compact convex set C contained in B. From B_i, we can select the members as the minimal cover of C, Where "covering" means that their union contains C; And "minimality" means that these members all intersect with C, removing these members and none of the remaining members of B intersect C.

For sequence $\{C_j\}_j$, We can find a subsequence $\{C_{1,j}\}_j$, making all members have consistent minimal coverage with respect to B_1. Among them, the writing order $C_{1,1}, C_{1,2}, \ldots$ is consistent with the relative order of these members in the $\{C_j\}_j$. For sequence $\left\{C_{1,j}\right\}_j$, We can find a subsequence $\left\{C_{2,j}\right\}_j$, making all members have consistent minimal coverage with respect to B_2 . Among them, the writing order $C_{2,1}, C_{2,2}, \ldots$ is consistent with the relative order of these members in the $\left\{C_{1,j}\right\}_j$. Step by step in this way. For each i , we find a subsequence $\left\{C_{i,j}\right\}_j$ from the sequence $\left\{C_{i-1,j}\right\}_j$ found in the previous step, making all members have consistent minimal coverage with respect to B_i , and the writing order $C_{i,1}, C_{i,2}, \ldots$ is consistent with the relative order of these members in $\left\{C_{i-1,j}\right\}_j$. From these sequences obtained step by step,

$$\begin{array}{lllll} C_{1,1}, & C_{1,2}, & \ldots & C_{1,j}, & \cdots \\ C_{2,1}, & C_{2,2}, & \ldots & C_{2,j}, & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ C_{j,1}, & C_{j,2}, & \cdots & C_{j,j}, & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{array}$$

We select members to form a new sequence. In particular, this sequence is $\left\{C_{i,i}\right\}_i$.

We proved that $\left\{C_{i,i}\right\}_i$ is the Cauthy sequence. To this end, the following facts are noted: for each i, j, k ,

$$\delta_H\left(C_{i,j}, C_{i,k}\right) \leq \frac{\sqrt{n}}{2^i} d \tag{12}$$

When $j \geq i$, for each k, l

$$\delta_H\left(C_{i,k}, C_{j,l}\right) \leq \frac{\sqrt{n}}{2^i} d \tag{13}$$

So, when $j \geq i$

$$\delta_H\left(C_{i,i}, C_{j,j}\right) \leq \frac{\sqrt{n}}{2^i} d \tag{14}$$

Theorem 2 The convergence $\lim_{i \to \infty} \delta_H\left(K_i, K\right) = 0$ is equivalent to the following conditions taken together

(1) each point in K is the limit of a sequence $\left(x_i\right)_{i \in N}$ with $x_i \in K_i$ for $i \in N$

(2) the limit of any convergent sequence $\left(x_{i_j}\right)_{j \in N}$ with $x_{i_j} \in K_{i_j}$ for $j \in N$ belongs to K

2 Proof of continuity of Gauss volume

Here, I solve the Gauss volume continuity from this point of view by combining the convergence property of Hausdorff metric with the explicit function.

Theorem 3 (continuity of Gauss volume) if $\lim_{i\to\infty}\delta_H\left(K_i,K\right)=0$, The corresponding indicative function converges. Thus, the Gauss volume is continuous.

Proof: Firstly, the explicit function of convex sequence K_i is given

$$I_{K_i}=\begin{cases}1 & x\in K_i\\ 0 & x\notin K_i\end{cases} \tag{15}$$

Then the explicit function of convex body K is given

$$I_K=\begin{cases}1 & x\in K\\ 0 & x\notin K\end{cases} \tag{16}$$

Then according to the definition of Hausdorff metric of convex sequence convergence, it can be obtained, when $i\to\infty$,

$$\begin{aligned}&K\subset K_i+\varepsilon B^n\\ &K_i\subset K+\varepsilon B^n\end{aligned} \tag{17}$$

The convergence equivalence conditions of convex body sequences mentioned earlier and the above inclusion relations, it can be concluded that when $i\to\infty$,for $\forall\varepsilon>0,\exists N\left(\varepsilon,x\right)$ for $n>N\left(\varepsilon,x\right)$,

$$\left|I_{K_i}-I_K\right|<\varepsilon \tag{18}$$

We can get that explicit functions sequence converges.

$$\text{Due to } \left|I_{K_i}\right|\le 1 \tag{19}$$

Combining the above two conclusions, it can be obtained from Lebesgue's control convergence theorem

$$\begin{aligned}&\lim_{i\to\infty}\gamma_n\left(K_i\right)=\lim_{i\to\infty}c_n\int_{K_i}e^{-\|x\|/2}I_{K_i}dx\\ &=c_n\int_{K_i}e^{-\|x\|/2}\lim_{i\to\infty}I_{K_i}dx\\ &=\gamma_n\left(\lim_{i\to\infty}K_i\right)=c_n\int_{K_i}e^{-\|x\|/2}I_Kdx\\ &=\gamma_n\left(K\right)\end{aligned} \tag{20}$$

3 Proof of the existence of Gauss John ellipsoid

We know that in the process of proving the existence of John ellipsoid, a very classical method is to construct a maximized sequence, and obtain the existence of John ellipsoid according to the properties of maximized sequence and the continuity and monotonicity of volume functional Therefore, this also gives us the enlightenment to solve the existence problem of Gauss John ellipsoid, which is proved as follows

Theorem 4 (existence of Gauss John ellipsoid) In the convex body K with symmetrical origin, there is an ellipsoid with the largest Gauss volume.

Proof: The problem is equivalent to a convex body K in R^n, finding an ellipsoid in ellipsoid sequence $\{E_n\}$ to solve the following maximization problem ($\{E_n\}$ is the ellipsoidal sequence in K)

$$\gamma_n(E_0) = \sup_{E_n} \gamma_n(E_n) \quad E_n \subseteq K \tag{21}$$

Due to $\gamma_n(E) = c_n \int_E e^{-\|x\|/2} dx < V(K) < +\infty$, it can be concluded that the maximum value $\sup_{E_n} \gamma_n(E_n)$ exists. Then the maximization sequence $\{E_n\}_n$ is constructed, making

$$\gamma_n(E_n) \le \gamma_n(E_{n+1}) \tag{22}$$

According to the monotone boundedness theorem

$$\lim_{n\to\infty} \gamma_n(E_n) = \sup_{E_n} \gamma_n(E_n) \tag{23}$$

Then by the Blaschke selection theorem given before, a convergent subsequence $\{E_{n_j}\}_j$ can be found, so we can get

$$\lim_{j\to\infty} \gamma_n(E_{n_j}) = \lim_{n\to\infty} \gamma_n(E_n) = \sup_{E_n} \gamma_n(E_n) \tag{24}$$

From the convergence of subsequences $\{E_{n_j}\}_j$, the Gauss volume continuity can be obtained according to Theorem 3, that is

$$\lim_{j\to\infty} \gamma_n(E_{n_j}) = \gamma_n(E_0) \tag{25}$$

Because of the compactness of convex body K, we finally get

$$E_0 \subset K \tag{26}$$

That is, there is an ellipsoid E_0 in the convex body K so that

$$\gamma_n\left(E_0\right)=\lim_{j\to\infty}\gamma_n\left(E_{n_j}\right)=\lim_{n\to\infty}\gamma_n\left(E_n\right)=\sup_{E_n}\gamma_n\left(E_n\right) \tag{27}$$

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