Plotting Lamb waves dispersion curves of an aluminum plate by the Semi-Analytical Finite Element (SAFE) method and comparison with analytical curves.

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Abstract. It is well known that the propagation of Lamb waves in elastic solid medias depends on the frequency and on the material properties, but it does not depend on the propagation direction in isotropic medias and gives rise to two basic types of modes, symmetric and antisymmetric ones. This paper presents an application of the semi-analytical finite element (SAFE) method on an isotropic aluminum plate of thickness *e = 2mm*. The main goal of this paper is to make a quantitative evaluation of the SAFE method by carrying out the following tasks: a) calculation of the dispersion curves (*wavenumber*, *phase velocity*, *group velocity*, b) calculation of the relative error and validation of the proposed algorithm, c) comparison between the curves obtained by SAFE method on those obtained by Bisection method. It should be noted here that our study is restricted on the propagative modes.

Keywords. Semi-Analytical Finite Element (SAFE) method, Bisection method, Lamb waves, Dispersion curves, Isotropic plates.

1 Introduction

There are several numerical methods for calculating the Lamb waves dispersion curves. Among these methods, there is the semi-analytical finite element (SAFE) method which is useful for composite and isotropic plates. Lamb waves propagation is multimodal and when there are several modes present for a given excitation frequency, the signal interpretation would be considerably difficult, hence the need to calculate the Lamb waves dispersion curves to limit the choice of excitation frequency to a domain not presenting many modes. In this paper, the application of the SAFE method is made on a homogeneous isotropic aluminum plate of thickness *e=2mm* with free boundaries. Our goal is to make a quantitative evaluation of the semi-analytical finite element (SAFE) method by doing the following tasks: a) calculation of the dispersion curves (*wavenumber*, *phase velocity*, *group velocity*, b) calculation of the relative error and validation of the proposed algorithm, c) comparison between the curves obtained by SAFE method on those obtained by a Bisection method. Our study was limited to propagative modes, purely imaginary or complex solutions are not considered here.

2 Analytic formulation of SAFE method

In previous studies, the authors discussed dispersion curves and their multiple applications in non-destructive testing. They propose several techniques for plotting dispersion curves. These methods include the Newton-Raphson method [1], bisection method [2], spectral method [3], transfer matrix method [4], and others [5]. Furthermore, dispersion curves are plotted by iterative methods, even though they are very close to the analytical curves. For this reason, we propose an alternative method that can better estimate the dispersion curves.

The iterative methods (bisection and Newton Raphson) are applicable only on isotropic plate. However, these methods are simple in terms of coding compared with the spectral and SAFE methods. Both last methods provide results of high accuracy with short time of calculation compared with the iterative methods for the composite and isotropic plate. The transfer matrix method can be more useful for multi-layered anisotropic media.

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Bartoli et al. [6] modeling Lamb wave propagation in arbitrary section waveguides using the SAFE method. Ahmad et al. [7] applied the method for plotting the dispersion curves of an aluminum plate and a composite structure.

Recently, Xing et al. [8] suggested a defect localization method for rails and plotted rail dispersion curves by the SAFE method. The technique for calculating pipeline dispersion curves immersed in a fluid using SAFE method has been presented by Wenbo et al. [9].

The SAFE method combines the FE method with analytical expressions. Using this method, Lamb wave propagation curves for isotropic and composite plates can be calculated. However, when using the SAFE method, we don't need to discretize the entire plate, it is enough to do for the cross-section of the plate.

The advantage of this solution over other iterative methods in the literature appears in the fact that the SAFE method doesn't only allow us to calculate the dispersion curves, but also the nodal displacements, and using interpolation one can easily calculate the deformations and the stresses.

We consider an infinitely wide plate in the *y*-axis direction as shown in figure 1, the cross-section domain of the plate is subdivided into a system of multilayers which are modeled using 1D finite elements with three-node as shown in figure 1. In our study, the wave propagates along the *z*-axis with a wavenumber *k* and at frequency *ω*.

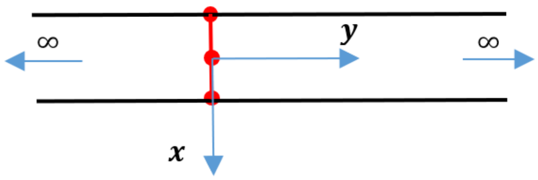

Fig. 1. Infinite (2D) plate with three nodes per element

The element displacement can be approximated by using the interpolation function $\mathbf{N}(x)$ and nodal unknown displacement $\mathbf{q}^{(e)}$ as follows

$$\mathbf{u}^{(e)}(x, z, t) = \mathbf{N}(x)\mathbf{q}^{(e)} e^{i(kz-\omega t)} \quad (1)$$

The element strain expression becomes

$$\varepsilon^{(e)} = \left[\mathbf{L}_x \frac{\partial}{\partial x} + \mathbf{L}_z \frac{\partial}{\partial z}\right] \mathbf{N}(x)\mathbf{q}^{(e)} e^{i(kz-\omega t)} = (\mathbf{B}_1 + ik\mathbf{B}_2)\, \mathbf{q}^{(e)} e^{i(kz-\omega t)} \quad (2)$$

where

$$\boldsymbol{B}_1 = \boldsymbol{L}_x \boldsymbol{N}_{,x} \ , \ \boldsymbol{B}_2 = \boldsymbol{L}_z \boldsymbol{N}$$

and $\boldsymbol{N}_{,x}$ is the derivative of $\boldsymbol{N}$ concerning *x*.

with

$$\mathbf{L}_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{L}_z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The equation of motion for the cross-section can be obtained by using Hamilton's principle:

$$[\boldsymbol{K}_1 + ik\boldsymbol{K}_2 + k^2\boldsymbol{K}_3 - \omega^2\boldsymbol{M}]\boldsymbol{U} = 0 \quad (3)$$

where

$$\boldsymbol{K}_1 = \cup_{e=1}^{n} \mathbf{k}_1^{(e)} \quad (4) \qquad \boldsymbol{K}_2 = \cup_{e=1}^{n} \mathbf{k}_2^{(e)} \quad (6)$$

$$\boldsymbol{K}_3 = \cup_{e=1}^{n} \mathbf{k}_3^{(e)} \quad (5) \qquad \boldsymbol{M} = \cup_{e=1}^{n} \mathbf{m}^{(e)} \quad (7)$$

and *n* denotes the number of elements with

$$\mathbf{k}_1^{(e)} = \int_x \mathbf{B}_1^T \mathbf{C}_e \mathbf{B}_1 \, dx \quad (8)$$

$$\mathbf{k}_2^{(e)} = \int_x (\mathbf{B}_1^T \mathbf{C}_e \mathbf{B}_2 - \mathbf{B}_2^T \mathbf{C}_e \mathbf{B}_1) \, dx \quad (9)$$

$$\mathbf{k}_3^{(e)} = \int_x \mathbf{B}_2^T \mathbf{C}_e \mathbf{B}_2 \, dx \quad (10)$$

$$\mathbf{m}^{(e)} = \int_x \mathbf{N}^T \boldsymbol{\mu}_e \mathbf{N} \, dx \quad (11)$$

where $\mathbf{C}_e$ and $\boldsymbol{\mu}_e$ are the elastic stiffness tensor and the density for the generic element, respectively.

The integrals given above are approximated by using Gaussian quadrature integration.

The global matrices $\mathbf{K}_1$, $\mathbf{K}_3$, and $\mathbf{M}$ are symmetric. However, the global matrix $\mathbf{K}_2$ is skew-symmetric. To simplify the equation numerical resolution, $\mathbf{K}_2$ is transformed into a symmetric form by pre-multiplying and post-multiplying equation with transformation matrix $\mathbf{T}^\mathbf{T}$ and $\mathbf{T}$ respectively.

$$\begin{cases} \mathbf{K}_1 = \mathbf{T}^\mathbf{T}\mathbf{K}_1\mathbf{T} \\ \widehat{\mathbf{K}_2} = i\mathbf{T}^\mathbf{T}\mathbf{K}_2\mathbf{T} \\ \mathbf{K}_3 = \mathbf{T}^\mathbf{T}\mathbf{K}_3\mathbf{T} \\ \mathbf{M} = \mathbf{T}^\mathbf{T}\mathbf{M}\mathbf{T} \\ \widehat{\mathbf{U}} = \mathbf{T}^\mathbf{T}\mathbf{U} \end{cases} \quad (12)$$

$\mathbf{T}$ is the transformation matrix with each diagonal element equal to 1, except every second (corresponding to the displacement in the **z**-direction) being equal to *i*.

As an alternative approach, the equation (3) can be reformulated as follows

$$\left[\mathbf{K}_1 + k\widehat{\mathbf{K}_2} + k^2\mathbf{K}_3 - \omega^2\mathbf{M}\right]\widehat{\mathbf{U}} = 0 \quad (13)$$

where $\widehat{\mathbf{K}_2}$ and $\widehat{\mathbf{U}}$ are respectively the $\mathbf{K}_2$ symmetric matrix and the new eigenvector.

The equation of motion can be solved by using two approaches. The first one can be applied by fixing the wavenumber k and searching for the frequency ω. The second one is proceeded by fixing the frequency ω and searching for the positive wavenumber k. It should be noted that the last approach which is used in this paper.

In this approach, equation (13) is simplified as linear form [10]

$$(\mathbf{A}-k\mathbf{B})\mathbf{Q} = \mathbf{0} \qquad (14)$$

where

$$\mathbf{A}=\begin{bmatrix} 0 & \boldsymbol{K_1}-\omega^2\boldsymbol{M} \\ \boldsymbol{K_1}-\omega^2\boldsymbol{M} & \widehat{\boldsymbol{K_2}} \end{bmatrix},\ \mathbf{B}=\begin{bmatrix} \boldsymbol{K_1}-\omega^2\boldsymbol{M} & 0 \\ 0 & -\boldsymbol{K_3} \end{bmatrix}$$

and

$$\mathbf{Q}=\begin{bmatrix} \widehat{\boldsymbol{U}} \\ k\widehat{\boldsymbol{U}} \end{bmatrix}.$$

It should be noted that all the eigenvalues and eigenvectors are real. The eigenvalues represent the wavenumbers for all the propagating modes, while the corresponding eigenvectors represent the cross-sectional displacement.

From a coding point of view, during the research of the solutions which are in this case the wavenumbers k, it is necessary to create a matrix which will be filled by these solutions. Whereas among these solutions one can have complex or pure imaginary solutions or negative real values and since we are interested only by positive solutions one eliminates any undesirable solution by well-defined logical instructions.

The wavenumbers obtained by solving the eigenvalue problem can be used to calculate the phase velocity:

$$V_p = \omega \,/\, k \qquad (15)$$

By simple manipulation, equation (14) leads to calculating the group velocity using the following relation

$$V_g = \partial\omega/\partial k = \left(\boldsymbol{Q}_{\boldsymbol{R}}^{T} K'\boldsymbol{Q}_{\boldsymbol{R}}\right)/\left(2\omega\boldsymbol{Q}_{\boldsymbol{R}}^{T}\boldsymbol{M}\boldsymbol{Q}_{\boldsymbol{R}}\right) \qquad (16)$$

where $\boldsymbol{Q}_{\boldsymbol{R}}$ indicates the right eigenvector of the system define by

$$\boldsymbol{Q}_{\boldsymbol{R}} = \widehat{\mathbf{U}}$$

and

$$\mathbf{K}' = \widehat{\mathbf{K}}_2 + 2k\mathbf{K}_3\,.$$

3 Proposed Algorithm

An algorithm is proposed as shown in figure 2 to plot the dispersion curves:

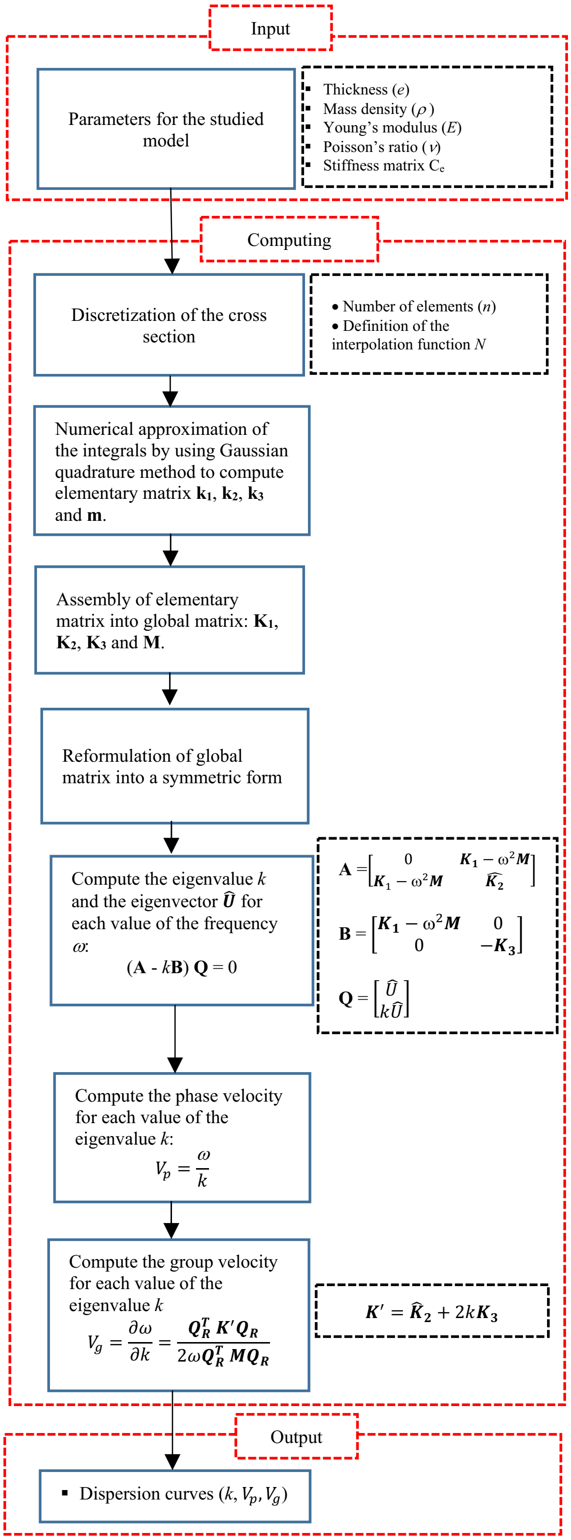

Fig. 2. Algorithm to plot the dispersion curves.

4 Dispersion curves

A Matlab code has been developed for plotting the dispersion curves. The wavenumber, phase velocity, and group velocity are plotted as a function of the frequency-thickness product (*fe*).

Let's take an aluminum plate with thickness *2mm*, Young's modulus $E = 70$GPa, Poisson's ratio $\nu = 0.32$, and mass density $\rho = 2700$ kg/m^3.

The minimum number of elements required to ensure the appearance of all modes in the range of frequencies between 0 and 5000 kHz [11]

$$n_{min} = (\beta fe/V_T) \qquad (17)$$

Here V_T is the transverse velocity of the propagative wave**,** and β depends on the order of the interpolation function.

In our case, we have used the quadrature interpolation function, thus $\beta = 4$ and $V_T = 3134\, m/s$. It follows that

$$n_{min} = 13.$$

In our case, we have taken *n =18* elements to compute the dispersion curves.

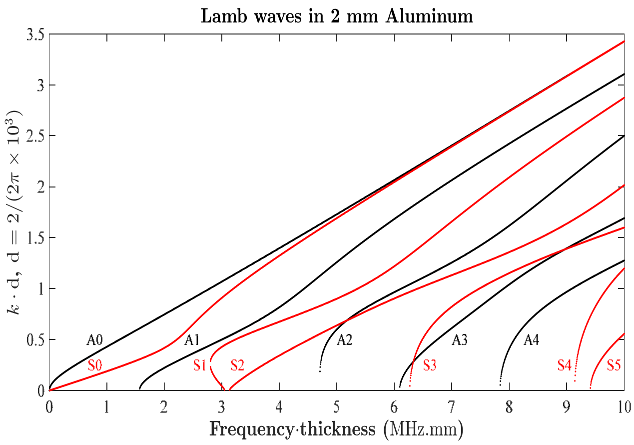

Fig. 3. Wavenumber curves for isotropic aluminum plate, *n* is taken to be equal to 18.

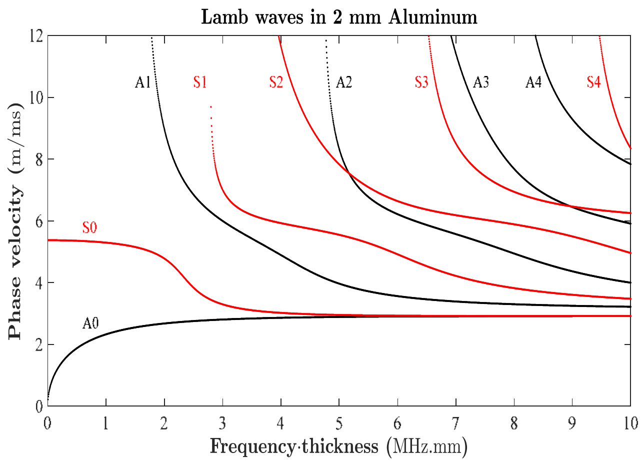

Fig. 4. Phase velocity curves for isotropic aluminum plate, *n* is taken to be equal to 18.

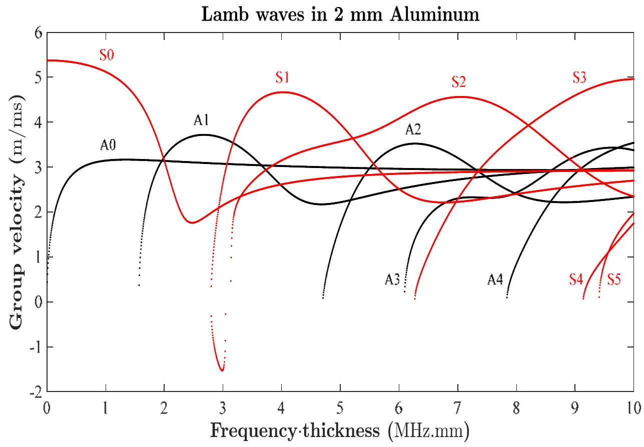

Fig. 5. Group velocity curves for isotropic aluminum plate, *n* is taken to be equal to 18.

The curves in figure 3 represent the wavenumber evolution of propagating modes versus the product frequency-thickness**.** For a given product (*fe)*, the modes present in the plate are known due to the wavenumbers. The wavenumber of the first symmetric modes (S0, S1, S2, S3, and S4) and antisymmetric modes (A0, A1, A2, A3, and A4) are plotted in figure 3.

The curves in figure 4 represent the phase velocity evolution of propagating modes versus the product frequency-thickness. The knowledge of phase velocity is essential when modeling the propagation of Lamb waves. Indeed, to generate a mode of Lamb at a given frequency-thickness product, we must know exactly the phase velocity with which the mode propagates in the plate to avoid that it crosses with another mode other than the desired mode. The phase velocity of the first symmetric modes (S0, S1, S2, S3, and S4) and antisymmetric modes (A0, A1, A2, A3, and A4) are plotted in figure 4.

The curves in figure 5 represent the group velocity evolution of propagating modes versus the product frequency-thickness. The group velocity is the velocity with which a wave packet propagates along with the plate. The group velocity of the first symmetric modes (S0, S1, S2, S3, and S4) and antisymmetric modes (A0, A1, A2, A3, and A4) are plotted in figure 5.

5 Results and discussions

To plot the curves dispersion of homogeneous isotropic plate, we have developed an algorithm based on the SAFE method, then it has been transformed into code in Matlab software.

This method's precision of the solutions depends on the number of elements.

Table 1 illustrates that the average relative error for the two fundamental modes S0 and A0 by the SAFE method is a decreasing function of the number of elements.

Here we give the formula used to calculate the relative error and its average:

$$err = \left|\frac{k_{safe} - k_{ana}}{k_{ana}}\right| \times 100 \qquad (18)$$

where k_{safe} is the wavenumber calculated by the SAFE method and k_{ana} is the analytical wavenumber. The average relative error is calculated by

$$err_a = \sum_{j=1}^{m} err_j / m \qquad (19)$$

here m is the number of the considered solutions and is equal in our case to 40.

Table 1. The average relative error for modes S0 and A0 by the SAFE method

Number of elements (n)	*Average relative error (S0)*	*Average relative error (A0)*
1	6.82 %	4.01 %
4	1.42 %	1.36 %
7	0.47 %	0.45 %
10	0.17 %	0.17 %
13	0.074 %	0.073 %
14	0.057 %	0.056 %

To validate the proposed algorithm, we proceed by calculating the relative error between the values of the wavenumbers obtained by the SAFE method compared to those obtained by the bisection method.

It should be noted here that the relative error on the wavenumber for the minimum number of elements (*n = 13*) increases as an exponential function as shown in figure 6. The error increases as the frequency increases.

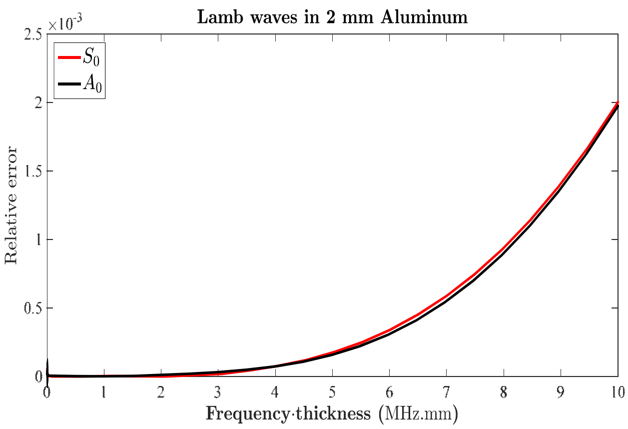

Fig. 6. Relative error on the wavenumber for *n = 13*

On the other hand, the dispersion curves obtained are compared with those calculated by the bisection method. The results are shown in figure 7, and one can see a good convergence of the SAFE method to the analytical curves (bisection method).

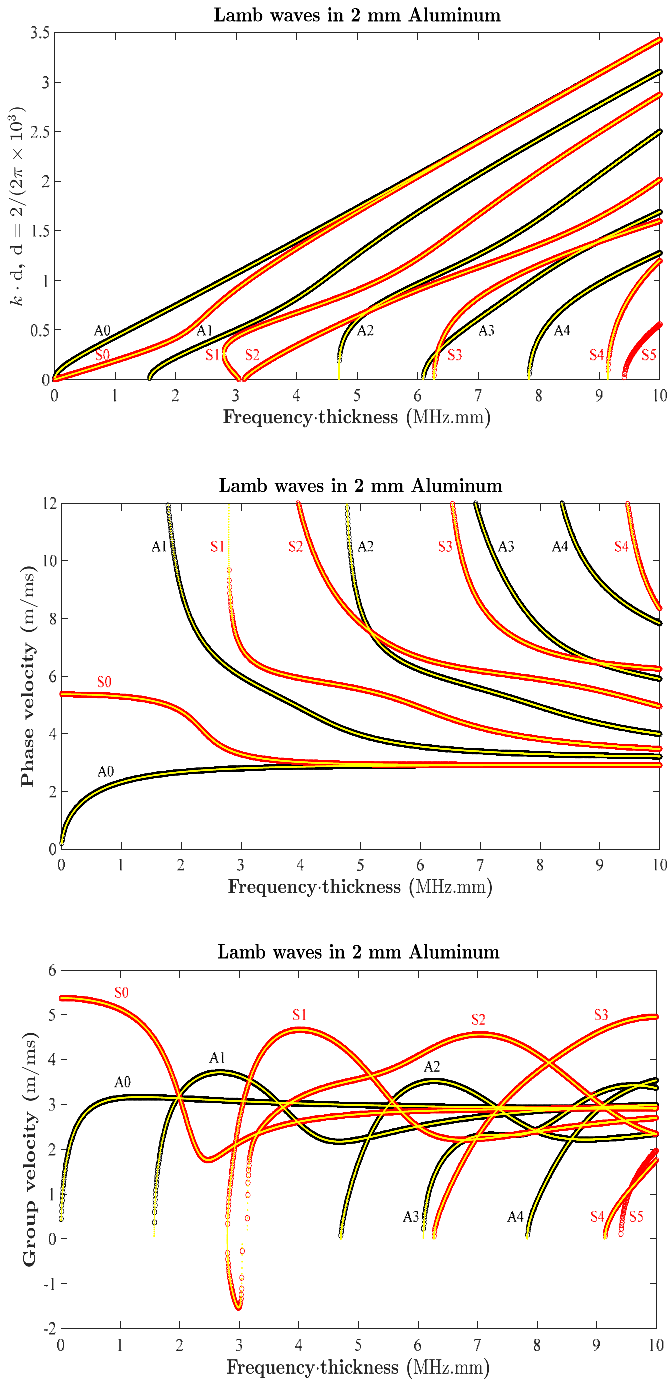

Fig. 7. Dispersion curves obtained by the SAFE method superposed on the analytical curves (yellow) for n = 18.

6 Conclusion

In this paper, the SAFE method is applied to calculate propagating modes with higher precision and with a large number of elements within plate thickness with a great frequency range. The results show a very good agreement between the results obtained by the code developed using the SAFE method and those obtained by using the bisection method. In addition, the bisection method is simple in terms of coding compared to the

SAFE method. However, the SAFE method can be more useful for composite or multilayer materials.

It should also be mentioned that the dispersion curves obtained by SAFE method are crucial for any theoretical, numerical or experimental study and for detecting damage in industrial structures.

The SAFE method doesn't only allow us to calculate the dispersion curves, but also the nodal displacements, and using interpolation one can easily calculate the deformations and the stresses. This purpose can be the subject of a future paper.

As perspective, it is planned to apply the SAFE method in industrial case.

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