Semi-Analytical Finite Element (SAFE) method for plotting Lamb waves dispersion curves of an aluminum plate and comparison with Disperse software.

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Abstract. Investigation into ultrasonic Non-destructive evaluation (NDE) methods for assessing complex designs depends firmly on displaying apparatuses that calculate dispersion curves. These expectations are vital for empowering the best assessment methodologies to be distinguished and their aversions to being assessed. This study, the main purpose of this work is to do a quantitative evaluation of the SAFE technique by calculating dispersion curves (wavenumber, phase velocity, and group velocity) and comparing them to those generated by Disperse Software.

Keywords. Semi-Analytical Finite Element (SAFE), Disperse software, Lamb waves, Dispersion curves

1 Introduction

Lately, there is a developing interest in the use of elastic-guided waves in non-destructive testing (NDT). These waves show up in structures wherever one aspect is smaller compared to the others, concerning model wires, pipes, bars, plates, or shells [1], [2].

The wavelength, dispersivity, phase, and group velocities of guided waves are all displayed on dispersion curves at a specific frequency [3]. When performing inspection measures, it's helpful to create a visual model of the structure in question. This often entails determining the theoretical wave motion modality and dispersivity for plates and other waveguides.

These relationships, however, do not hold for increasingly complicated geometries and material models. Various matrix algorithms were developed to describe guided waves in layered media [4]. These techniques have been applied to a wide range of applications from thin film coatings to global seismology. These techniques were used to develop the software Disperse [5], a widely used research tool for plate modeling.

The authors have previously studied dispersion curves and their various uses in NDT. They proposed several iterative methods for plotting dispersion curves.

However, even though the dispersion curves generated by iterative approaches are very close to analytical curves, they still show errors in calculating the cutoff frequencies. As a result, we suggest an alternative strategy for a better dispersion curve estimate of the solution [6].

Many researchers utilized the SAFE method to examine Lamb wave propagation in isotropic and composite structures. The method was used by Ahmad et al. [7] to plot the dispersion curves of an aluminum plate and composite construction. The SAFE approach was used by Bartoli et al. [8] to model the wave propagation of Lamb waves in waveguides of any section.

Wenbo et al. [9] recently published a methodology for estimating pipeline dispersion curves immersed in a fluid that incorporates the SAFE method approach and the perfect match layer technique. The SAFE method was used by Hayashi et al [10]. to plot rail dispersion curves for railroads.

Computer models and visualization have also revealed sophisticated directed wave movements. Particular calculation techniques were developed for these numerical analyses to express guided wave propagation

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over long distances, such as the hybrid method and the finite element method (FEM) or boundary element method (BEM), and the semi-analytical finite element (SAFE) approach, which treats wave motions in the propagation direction theoretically and subdivides the cross-section [11].

The SAFE approach is used to simulate guided-wave propagation of homogeneous isotropic aluminum plate with free borders in this study. Analytically, the displacements along the wave propagation direction can be characterized as harmonic exponential functions. Only the cross-section needs to be discretized using this procedure[12] [8], by constructing a large number of large complex stiffness matrices [13]. The SAFE codes generate the solutions from an eigenvalue problem, they do not require the root finding procedures utilized in iterative approaches. Our goal is to do an evaluation of the semi-analytical finite element (SAFE) and compare the solutions to those obtained by the Disperse Software.

2 SAFE Method

Lamb wave propagation curves for isotropic and composite plates can be determined using this method. When solving the dispersive relation using SAFE, The methodology includes both FE and analytical formulations. Only the cross-section of the plate is discretized with Finite Element (FE). The simulated displacements are using analytical harmonic functions in the wave propagation direction, i.e. normal to the cross-section plane or line, hence the name semi-analytical [14]. This offers computational savings, especially for short wavelengths when a very thin mesh is required [15].

In the SAFE method, at every frequency, a discrete number of directed modes is acquired. The waveguide cross-section is placed on the x-y plane for a Cartesian reference framework (See Fig.1).

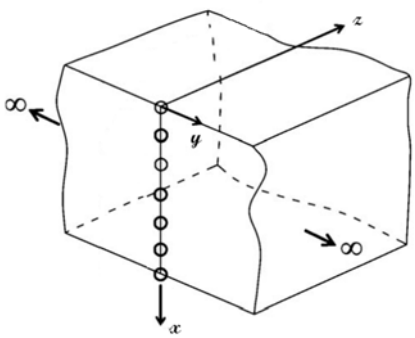

Fig. 1. Cross-section of a plate waveguide in Cartesian coordinates for SAFE

The harmonic motion is represented by introducing a harmonic displacement field along the propagation direction, z, and interpolating the displacement inside the element in the cross-sectional plane x-y [8].

The displacement field's final expression is:

$$\mathbf{U}(x,y,z,t)=\begin{bmatrix}u(x)\\ w(x)\end{bmatrix}e^{i(kz-\omega t)} \qquad (1)$$

where $i^2=-1$, ω is the angular frequency and k is the wavenumber.

It can be observed that by assuming an exponential harmonic term for the displacement field in (z, t) for an infinite plate in y, the x dependence can be described using the interpolation functions with discrete technique[14].

The plate cross-section domain is partitioned into a multilayer system[16]. Interpolation functions matrix **N**(x) and nodal unknown displacements can be used to approximate the element displacement. For a three-node one-dimensional element, the displacement field can be expressed as:

$$\mathbf{U}^{(\mathbf{e})}(x,z,t)=\mathbf{N}(x)\mathbf{q}^{(\mathbf{e})}e^{i(kz-\omega t)} \qquad (2)$$

The element strain can be expressed as:

$$\boldsymbol{\varepsilon}^{(e)}=\left[\mathbf{L_x}\frac{\partial}{\partial x}+\mathbf{L_z}\frac{\partial}{\partial z}\right]\mathbf{N}(x)\mathbf{q}^{(\mathbf{e})}e^{i(kz-\omega t)}$$

$$=(\mathbf{B_1}+ik\mathbf{B_2})\,\mathbf{q}^{(\mathbf{e})}e^{i(kz-\omega t)} \qquad (3)$$

in which $\mathbf{B_1}=\mathbf{L_x}\mathbf{N}_{,\mathbf{x}}$ (4) $\mathbf{B_2}=\mathbf{L_z}\mathbf{N}$ (5)

$\mathbf{N}_{,\mathbf{x}}$ is the derivative of the shape functions in *x*.

The quadratic system of eigenvalue equations is obtained by substituting these functions into the equation of motion and building the elementary matrices by using the Hamilton principle:

$$[\mathbf{K_1}+ik\mathbf{K_2}+k^2\mathbf{K_3}-\omega^2\mathbf{M}]\mathbf{U}=0 \qquad (6)$$

where **U** is the vector of amplitudes of the global displacement field; the matrices $\mathbf{K_1}$, $\mathbf{K_2}$, $\mathbf{K_3}$, and $\mathbf{K_4}$ are defined by

$$\mathbf{K_1}=\cup_{e=1}^{nel}\mathbf{k_1^{(e)}} \quad (7) \qquad \mathbf{K_2}=\cup_{e=1}^{nel}\mathbf{k_2^{(e)}} \quad (8)$$

$$\mathbf{K_3}=\cup_{e=1}^{nel}\mathbf{k_3^{(e)}} \quad (9) \qquad \mathbf{M}=\cup_{e=1}^{nel}\mathbf{m^{(e)}} \quad (10)$$

with

$$\mathbf{k_1^{(e)}}=\int_x^0\mathbf{B_1^T}\mathbf{C_e}\mathbf{B_1}\,dx \qquad (11)$$

$$\mathbf{k_2^{(e)}}=\int_x^0(\mathbf{B_1^T}\mathbf{C_e}\mathbf{B_2}-\mathbf{B_2^T}\mathbf{C_e}\mathbf{B_1})\,dx \qquad (12)$$

$$\mathbf{k_3^{(e)}}=\int_x^0\mathbf{B_2^T}\mathbf{C_e}\mathbf{B_2}\,dx \qquad (13)$$

$$\mathbf{m^{(e)}}=\int_x^0\mathbf{N^T}\boldsymbol{\mu}_\mathbf{e}\mathbf{N}\,dx \qquad (14)$$

$\mathbf{C_e}$ and μ_e are the elastic stiffness tensor and the density for the generic element, respectively.

$\mathbf{K_2}$ is changed into a symmetric form by pre-multiplying and post-multiplying equations using transformation matrices $\mathbf{T^T}$ and $\mathbf{T}$, respectively, to simplify computational savings' numerical resolution.

$$\mathbf{T^T K_1 T = K_1} \quad (15) \qquad \mathbf{T^T K_3 T = K_3} \quad (16)$$
$$\boldsymbol{T^T K_2 T = -i\widehat{K_2}} \quad (17) \qquad \mathbf{T^T M T = M} \quad (18)$$
$$\mathbf{T^T U = \widehat{U}} \quad (19)$$

The equation (6) can also be rewritten as:

$$\left[\mathbf{K_1} + k\widehat{\mathbf{K_2}} + k^2\mathbf{K_3} - \omega^2\mathbf{M}\right]\widehat{\mathbf{U}} = 0 \quad (20)$$

The $\mathbf{K_2}$ symmetric matrix and the new eigenvector are represented by $\widehat{\mathbf{K_2}}$ and $\widehat{\mathbf{U}}$, respectively.

The integrations over the elements were computed using the Gauss quadrature rule in this study [17]. Fixing the angular frequency and solving the problem

The equation of motion can be solved by fixing the frequency ω and solving for the positive wavenumber *k*.

The answer can be written as an eigensolution, with the eigenvalues and eigenvectors having to satisfy certain conditions:

$$\left(\begin{bmatrix} 0 & \mathbf{K_1} - \omega^2\mathbf{M} \\ \mathbf{K_1} - \omega^2\mathbf{M} & \widehat{\mathbf{K_2}} \end{bmatrix} - k\begin{bmatrix} \mathbf{K_1} - \omega^2\mathbf{M} & 0 \\ 0 & -\boldsymbol{K_3} \end{bmatrix}\right)\begin{bmatrix} \widehat{\mathbf{U}} \\ k\widehat{\mathbf{U}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(21)

The global matrices $\mathbf{K_1}$, $\mathbf{K_3}$, and $\mathbf{M}$ are symmetric. However, the global matrix $\mathbf{K_2}$ is skew-symmetric.

The phase velocity can be calculated using the wavenumbers obtained by solving the eigenvalue problem:

$$V_p = {}^{\omega}/_{k} \quad (22)$$

This leads to the calculation of group velocity using the relation after some algebra manipulation[14].

$$V_g = \frac{\partial\omega}{\partial k} = \frac{\mathbf{Q_R^T\, K' Q_R}}{2\omega\mathbf{Q_R^T\, M Q_R}} \quad (23)$$

where the right eigenvector of the system is denoted by $\mathbf{Q_R}$.

and $$\mathbf{K'} = \widehat{\mathbf{K}}_2 + 2k\mathbf{K_3} \quad (24)$$

Proposing a real positive and solving for the wavenumber *k*, as well as proposing a real or complex wavenumber and solving for the angular frequency, can lead to an eigensolution. Typically, one offers an angular frequency and solves for the wavenumber $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{\omega})$, which allows the analysis of both propagating and non-propagating waves **[18]**.

3 Numerical results

In this part, a numerical example demonstrating the validity and correctness of the proposed SAFE for the analysis of guided waves in elastic-free plate problems is shown. To begin, these results are compared to numerical data acquired using the usual SAFE approach and analytical results generated using the software Disperse.

The SAFE formulation has been validated for a homogeneous aluminum plate of 2mm thickness with the following isotropic elastic characteristics: density $\rho = 2700\ kg/m3$, longitudinal velocity$V_L = 6091\mathrm{m/s}$, and shear velocity V_T = *3134 m/s*.

By solving the eigenvalue problem established in the previous section, the SAFE formulation may produce dispersion curves for Lamb waves in a simple manner. There are two methods to go through frequency sweep or wavenumber sweep.

The modeling method is separated into many sections, as shown in Fig. 2. The first step is to define the structure's conceptual model, as well as select a material model for the structure. The second step, create a FE model of the conceptual model and solve the eigenvalue problem that results. Finally, the collected solution is post-processed to save useful data. Physically meaningful roots are chosen, whereas false roots are rejected.

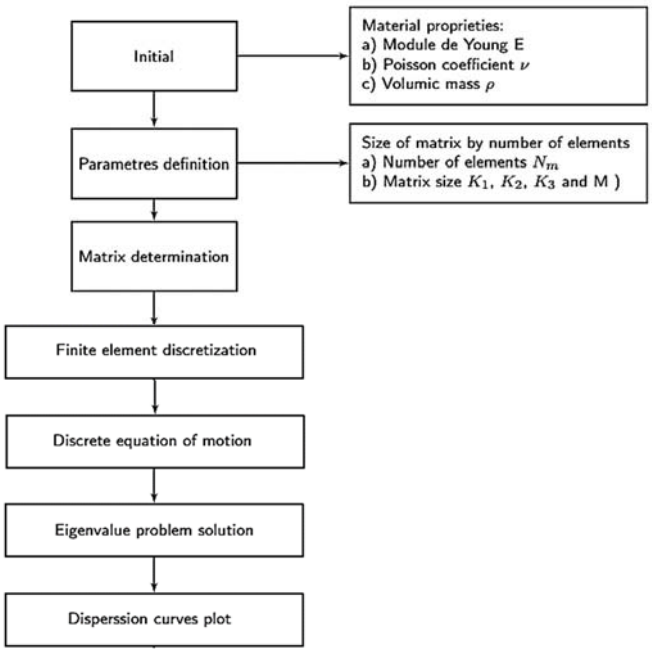

Fig. 2. Flowchart for the modeling process

3.1 Impact of number of elements

To more understand the impact of the number of elements. This last option is being considered as a variant in this research to see how it affects the results [19].

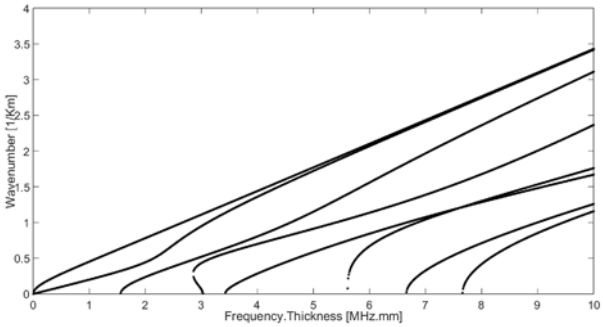

Fig. 3. The wavenumber dispersion curves by the SAFE method for *N=2*

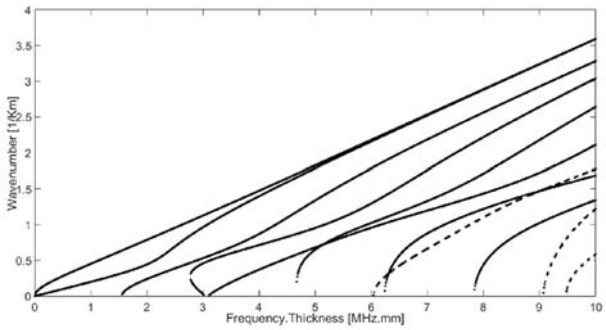

Fig. 4. The wavenumber dispersion curves by the SAFE method for *N=8*

The wavenumber curves generated by the SAFE approach with different numbers of elements *N=2*, and *N=8* are shown in Fig. 3 and Fig 4. Which illustrate the apparition of new modes.

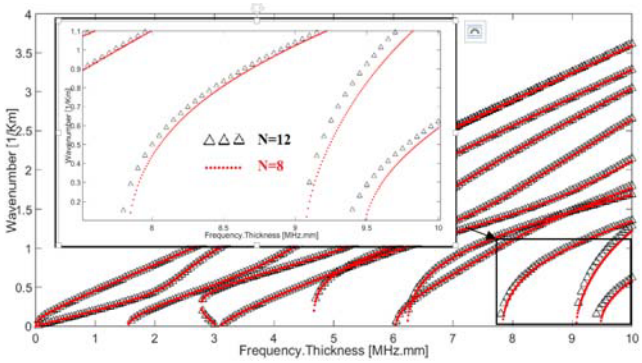

Fig. 5. Superposed wavenumber curves for different numbers of elements by the SAFE method

Comparing dispersion curves of a different numbers of elements Fig.5 justifies that the application of a lower number of elements in the input, it may not be capable to detect some roots, or, not achieving the desired precision, especially for the high product of thickness and frequency.

3.2 Minimum number of elements N_{min}

Some calculations may reduce precision as a result of speed optimizations performed by the compiler, the main factor is the number of subdivisions in the SAFE method related to the number of elements.

The following meshing criterion is provided based on the examination of the element stiffness matrix error **[20]**.

$$\frac{\lambda_T}{l} = \beta \qquad (25)$$

where l denotes the element's length, $\lambda_T = \frac{V_T}{f}$ denotes the wavelength of transverse waves propagating at velocity V_T, and $\beta = 4$ signifies the degree of the polynomial interpolation.

replacing l with e/N, where N is the number of elements the formulation in inequality is

$$\frac{NV_T}{fe} > \beta \qquad (26)$$

As a result, the required minimum number of elements can be calculated. to ensure the SAFE method's accuracy for a given product $F = f\,e$.

$$N_{min} = \left|\frac{\beta}{V_T}F\right| \qquad (27)$$

By replacing $\beta = 4$ and $V_T = 3134\ m/s$ in our example, and F = 10 MHz.mm, the minimal number of elements is: $N_{min} = 13$.

The application of this number of elements in a calculation does not ensure the desired precision.

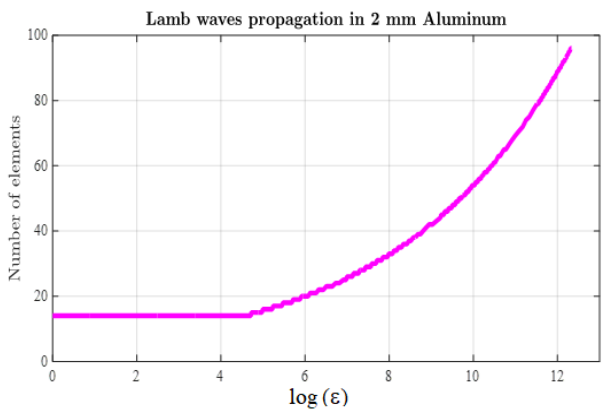

Fig. 6. Evolution of element number vs. precision (ε)

The precision of roots from the SAFE technique is shown against the number of elements in Fig. 6. The curve describes that to obtain a high precision an important number of elements is required.
Only the real roots of the dispersion relations that correspond to Lamb's propagating modes are taken into account in this study. The non-propagating (pure imaginary wavenumber) or attenuated (complex wavenumber) modes are not taken into account [17].

To compute the dispersion curves in our example, the number of elements used is $N = 18$ elements.

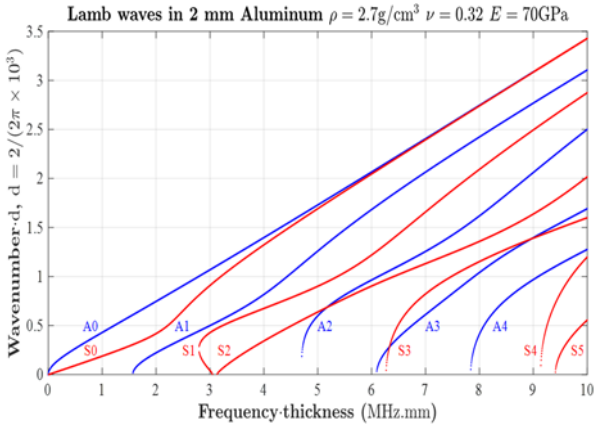

Fig. 7. Wavenumber curves for isotropic aluminium plate

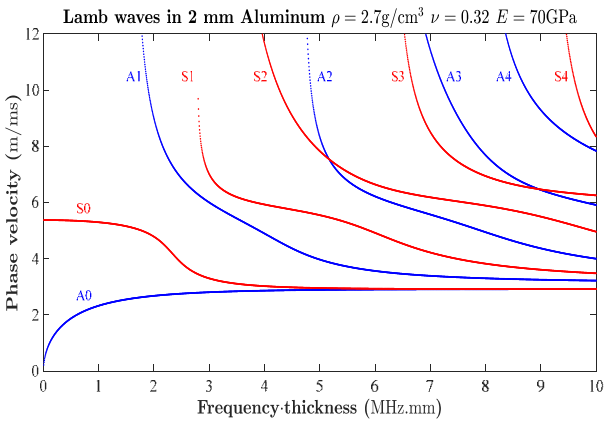

Fig. 8. Phase velocity curves for isotropic aluminium plate

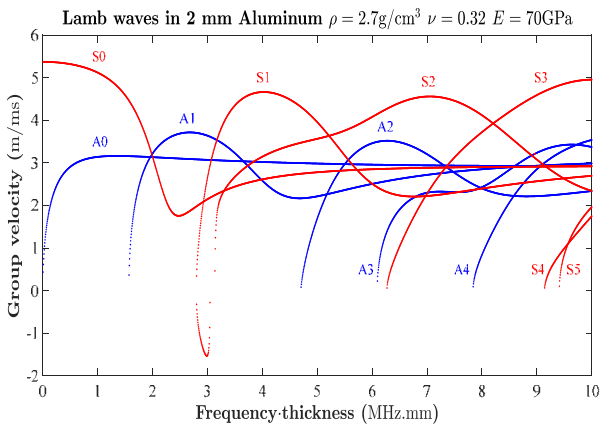

Fig. 9. Group velocity curves for the isotropic aluminium plate

Fig.7 shows the evolution of propagating wavenumber vs. the product frequency-thickness. The wavenumbers reveal the modes present in the plate for a specific product (f.e).

Fig.7 shows the wavenumbers of the first symmetric (S0, S1, S2, S3, and S4) and antisymmetric (A0, A1, A2, A3, and A4) modes.

Fig. 8 shows the phase velocity evolution of propagating modes vs. the frequency-thickness product. When modeling the propagation of Lamb waves, the understanding phase velocity is crucial. Indeed, to generate a Lamb mode at a certain frequency-thickness product, we must know the exact phase velocity with which the mode propagates in the plate so that it does not cross with another mode that is not the intended mode. Fig.8 shows the phase velocity of the first symmetric (S0, S1, S2, S3, and S4) and antisymmetric (A0, A1, A2, A3, and A4) modes.

Fig. 9 describes the evolution of propagating modes' group velocity vs. frequency-thickness product. The group velocity is the speed at which the wave packet containing the group pulse propagates along with the plate. Along with the plate, it also represents the wave's propagating energy speed. Fig.9 shows the group velocity of the initial symmetric modes (S0, S1, S2, S3, and S4), as well as the antisymmetric modes (A0, A1, A2, A3, and A4).

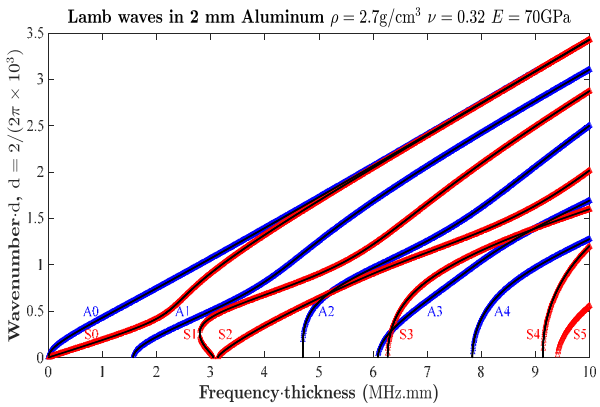

Fig. 10. Wavenumber curves superposed with Disperse

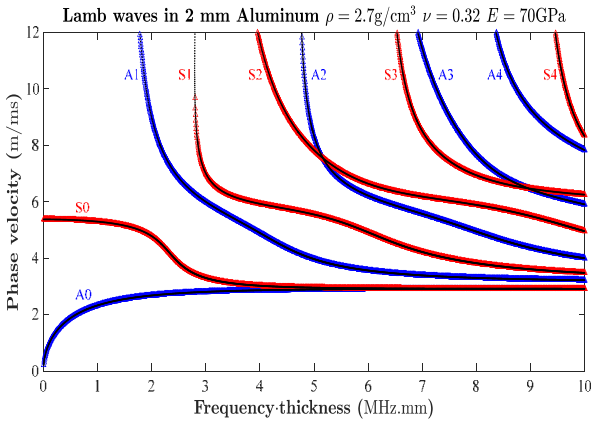

Fig. 11. Phase velocity curves superposed with Disperse

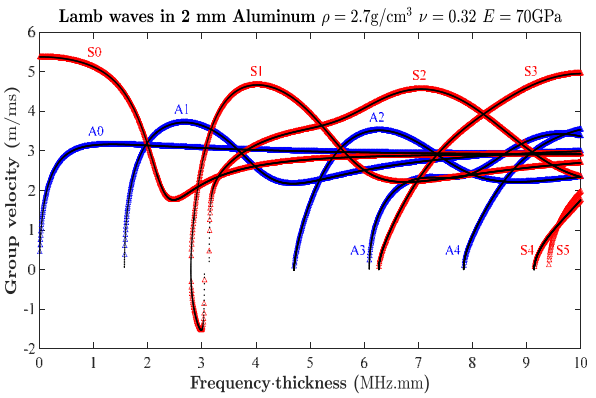

Fig 12. Group velocity curves superposed with disperse

The dispersion relations are accurately captured by the SAFE approach. The phase velocity of real-valued roots from the SAFE method is plotted against the Disperse dispersion curves for the five lowest modes, i.e. as shown in Fig.11. Along the dispersion curves, SAFE group velocity solutions can be discovered. For the five lowest modes, i.e. those illustrated in Fig.12, the wavenumber of real-valued roots from the SAFE technique is shown against the Disperse wave number in Fig. 10. As a result, due to the computing benefits, using a SAFE approach of cross-sections is recommended.

4 Conclusion

In this study, the SAFE dispersion results match those of the Disperse software for a continuous homogeneous isotropic linear elastic free plate. The results revealed that the method's accuracy is dependent on the number of meshing elements used. The number of elements per plate thickness must be tuned to achieve acceptable precision.

The approach proposed in this study can be used to investigate progressively the nodal displacement, deformation, and strain of a plate or complicated structures like composites or functionally graded material in the future.

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