The generalized semidiscrete cmKdV system and the periodic reduction

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Abstract. The complete integrability of a multicomponent differential-difference complex mKdV system with branched dispersion relation is proven. We use two approaches for this purpose. The first one is the Hirota bilinear formalism that helps us construct the multi-soliton solutions for a system of any M coupled equations. The same soliton solutions can be obtained through the periodic reduction approach, which has as a starting point a two-dimensional semidiscrete cmKdV equation. Plots of multi-solitons are also presented.

1 Introduction

The complete integrability of dynamical systems represent a topic of interest for the field of nonlinear science. There are several criteria and techniques for investigating integrability such as: complexity growth [1], singularity confinement [2], cube consistency [3], Lax representation [4–6], Lie symmetry approach [7, 8] and the Hirota bilinear and super-bilinear formalism [9–12]. For constrained systems and gauge theories, the Becchi-Rouet-Stora-Tyutin (BRST) technique [13–16] can be applied. The study of coupled discrete multicomponent systems integrability is a topic still not well developed. The Hirota formalism combined with the periodic reduction has proven to be very effective for these type of multicomponent systems, that have a very interesting phenomenology because of the extra degrees of freedom that arise from the matrix structure [17–19].

The subject of our investigation is a generalisation of the semidiscrete complex modified Korteweg–de Vries (cmKdV) equation to the multicomponent (matrix) case, with many branches of the dispersion relation. Several forms of semidiscrete cmKdV were analysed in literature during the last decades from different points of view [20–24] and important results were obtained: Lax pair, conservation laws, numerical simulations, N-fold Darboux transformation (DT), different types of exact solutions (anti-dark soliton, breather, periodic solutions, rogue wave solution). The starting point of our research is the very first semidiscrete version of cmKdV, proposed in 1976 by Ablowitz and Ladik [25] and recently solved via Hirota bilinear formalism in [26].

In this paper we go a step forward, analysing a multicomponent differential-difference cmKdV system with any M coupled equations with branched dispersion relation. Using the Hirota bilinear formalism for proving complete integrability, we find the N-soliton solution. The branched dispersion relation and the components' phases, parametrized by the order M roots of unity, are the main features of this system. The existence of many branches for the

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dispersion relation allows more freedom in soliton interaction. The paper is organised as follows: after a brief introduction given in Section 1, in Section 2 we present the general differential-difference cmKdV system with branched dispersion. In Section 3 we chose to discuss the particular case $M = 2$ and provide the Hirota bilinear form, the multi-soliton solutions and suggestive plots. In Section 4 we discuss the general case ($\forall$ M) and construct the N-soliton solution, proving complete integrability. In Section 5 we show that using the periodic reduction on a completely integrable two-dimensional semidiscrete cmKdV system, one can obtain the same multi-soliton solution. Last section is dedicated to conclusions.

2 The multicomponent differential-difference cmKdV system

Based on the first semidiscrete version of cmKdV [25], the general differential-difference cmKdV system with branched dispersion has the following form :

$$\frac{d}{dt}K_m = (1+\alpha K_m K_m^*)(E_{\sigma_1}K_{m+1}E_{\sigma_2} - E_{\sigma_2}K_{m-1}E_{\sigma_1}), \tag{1}$$

where $\alpha = \pm 1$ (the focusing/defocusing case for (+)/(-)), K_m is a diagonal matrix of complex functions $v_\mu(n,t)$, $\mu = 1,...,M$ (n the discrete space variable, t the continuous time variable):

$$K_m = \begin{pmatrix} v_1(n,t) & 0 & 0 & ...... & 0 \\ 0 & v_2(n,t) & 0 & ...... & 0 \\ .... & .... & .... & ..... & ...... \\ 0 & 0 & 0 & ...... & v_M(n,t) \end{pmatrix}$$

and $E_{\sigma_1}, E_{\sigma_2}$ are permutation matrices corresponding to the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & . & . & . & . & M \\ 2 & 3 & . & . & . & . & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & . & . & . & . & M \\ M & 1 & 2 & . & . & . & M-1 \end{pmatrix}.$$

On the components, system (1) has the following expression:

$$\begin{aligned} \dot{v}_1 &= (1+\alpha|v_1|^2)(\overline{v_2} - \underline{v_M}) \\ \dot{v}_2 &= (1+\alpha|v_2|^2)(\overline{v_3} - \underline{v_1}) \\ &..... \\ \dot{v}_{M-1} &= (1+\alpha|v_{M-1}|^2)(\overline{v_M} - \underline{v_{M-2}}) \\ \dot{v}_M &= (1+\alpha|v_M|^2)(\overline{v_1} - \underline{v_{M-1}}) \end{aligned} \tag{2}$$

where $v_\mu = v_\mu(n,t)$ and $\overline{v_\mu} = v_\mu(n+1,t), \underline{v_\mu} = v_\mu(n-1,t)$ for any $\mu = 1,...,M$.

For $M = 1$, system (2) reduces to the semidiscrete cmKdV equation investigated in [26]:

$$\dot{v}_1 = (1+\alpha|v_1|^2)(\overline{v_1} - \underline{v_1}). \tag{3}$$

For $M = 2$, we get the following system, which has two branches of dispersion:

$$\begin{aligned} \dot{v}_1 &= (1+\alpha|v_1|^2)(\overline{v_2} - \underline{v_2}) \\ \dot{v}_2 &= (1+\alpha|v_2|^2)(\overline{v_1} - \underline{v_1}). \end{aligned} \tag{4}$$

For $M = 3$, we get a multicomponent system with three branches of dispersion:

$$\begin{aligned} \dot{v}_1 &= (1+\alpha|v_1|^2)(\overline{v_2} - \underline{v_3}) \\ \dot{v}_2 &= (1+\alpha|v_2|^2)(\overline{v_3} - \underline{v_1}) \\ \dot{v}_3 &= (1+\alpha|v_3|^2)(\overline{v_1} - \underline{v_2}) \end{aligned} \tag{5}$$

and so on for any M.

3 The Hirota bilinear form and multi-soliton solutions for $M = 2$

The main goal of this paper is to investigate the integrability and the multi-soliton solutions for differential-difference cmKdV, the general case ($\forall M$). The concept of integrability is usually relying on the existence of an infinite number of independent integrals in involution that can be computed from the Lax pair. The Hirota bilinear formalism offers an alternative proof of integrability, more precisely it requires the existence of general multi-soliton solution that describes multiple collisions of an arbitrary number of solitons with arbitrary phases.

Considering the focusing case ($\alpha = 1$) of the semidiscrete cmKdV with two coupled equations (4), we use the nonlinear substitutions involving two tau functions each:

$$v_1 = G_1/F_1, \quad v_2 = G_2/F_2, \tag{6}$$

and cast (4) into the Hirota bilinear form:

$$\begin{aligned} \mathbf{D_t} G_1 \cdot F_1 &= \overline{G_2}\underline{F_2} - \underline{G_2}\overline{F_2} \\ \mathbf{D_t} G_2 \cdot F_2 &= \overline{G_1}\underline{F_1} - \underline{G_1}\overline{F_1} \\ F_1^2 + |G_1|^2 &= \overline{F_2}\underline{F_2} \\ F_2^2 + |G_2|^2 &= \overline{F_1}\underline{F_1} \end{aligned} \tag{7}$$

where F_1, F_2 are real functions, while G_1, G_2 are complex functions.

The anzatz for 1-ss of coupled focusing semidiscrete cmKdV has the form:

$$G_1 = e^{\eta_1}, \quad G_2 = \epsilon e^{\eta_1}, \quad F_1 = F_2 = 1 + e^{\eta_1+\eta_1^*+\psi_{12}}$$

where $\eta_1 = k_1 n + \omega_1 t + \eta_1^{(0)}$, k_1 is the wave number, ω_1 the angular frequency and $\eta_1^{(0)}$ an arbitrary phase.

The dispersion relation has two possible branches of dispersion:

$$\omega_1 = 2\epsilon \sinh(k_1), \quad \epsilon \in \{\pm 1\}$$

and the phase factor maintains the form as in the case of one component semidiscrete cmKdV (3) as calculated in [26]:

$$e^{\psi_{12}} = \frac{1}{2[\cosh(k_1 + k_1^*) - 1]}.$$

The 2-ss for two component semidiscrete cmKdV is:

$$\begin{aligned} G_1 &= e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+\eta_2^*+\psi_{12}+\psi_{14}+\psi_{24}} + e^{\eta_1+\eta_2+\eta_1^*+\psi_{12}+\psi_{13}+\psi_{23}} \\ G_2 &= \epsilon_1 e^{\eta_1} + \epsilon_2 e^{\eta_2} + \epsilon_1 e^{\eta_1+\eta_2+\eta_2^*+\psi_{12}+\psi_{14}+\psi_{24}} + \epsilon_2 e^{\eta_1+\eta_2+\eta_1^*+\psi_{12}+\psi_{13}+\psi_{23}} \\ F_1 &= 1 + e^{\eta_1+\eta_1^*+\psi_{13}} + e^{\eta_2+\eta_2^*+\psi_{24}} + e^{\eta_1+\eta_2^*+\psi_{14}} + e^{\eta_2+\eta_1^*+\psi_{23}} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\sum_{1\le i<j}^{4}\psi_{ij}} \\ F_2 &= 1 + e^{\eta_1+\eta_1^*+\psi_{13}} + e^{\eta_2+\eta_2^*+\psi_{24}} + \frac{\epsilon_1}{\epsilon_2} e^{\eta_1+\eta_2^*+\psi_{14}} + \frac{\epsilon_2}{\epsilon_1} e^{\eta_2+\eta_1^*+\psi_{23}} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\sum_{1\le i<j}^{4}\psi_{ij}} \end{aligned}$$

where:

$$\eta_j = k_j n + \omega_j t + \eta_j^{(0)}, \quad k_{j+2} = k_j^*, \quad \omega_j = 2\epsilon_j \sinh(k_j), \tag{8}$$

$$\epsilon_j \in \{\pm 1\}, \quad \epsilon_{j+2} = \epsilon_j, \quad j = 1, 2,$$

$$e^{\psi_{ij}} = \begin{cases} \frac{1}{2}\left[\epsilon_i\epsilon_j \cosh(k_i + k_j) - 1\right]^{-1}, & \text{if } i = 1,2 \text{ and } j = 3,4; \\ 2\left[\epsilon_i\epsilon_j \cosh(k_i - k_j) - 1\right], & \text{if } i = 1,2 \text{ and } j = 1,2; \\ & \text{or } i = 3,4 \text{ and } j = 3,4. \end{cases}$$

We present below several plots for the absolute value of the multi-soliton solution (8) of the focusing semidiscrete cmKdV for $M = 2$ given in (4), for some particular values and suitable intervals of n and t. More precisely, in Figure 1 one can see plots of the absolute value of the 2-soliton solution, while in Figure 2 the absolute value of the 3-ss, (9) for $N = 3$.

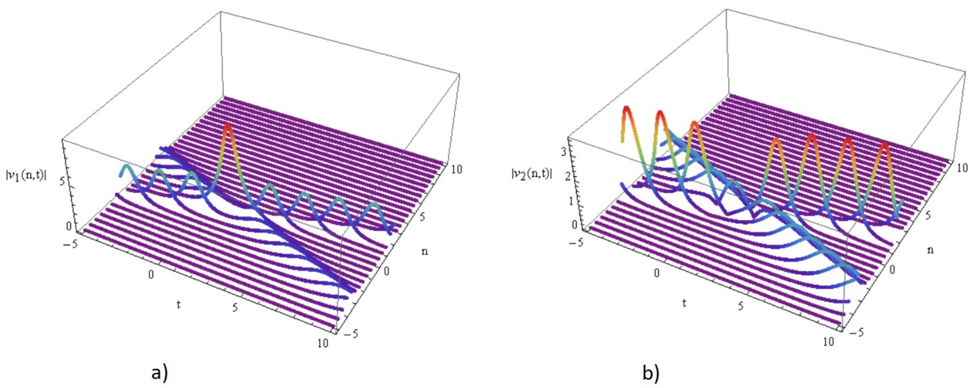

Figure 1. Plots of $|v_1(n,m)| = |G_1/F_1|$ (a) and $|v_2(n,m)| = |G_2/F_2|$ (b) given in (8), the absolute values of the 2-ss for the focusing semidiscrete cmKdV with two coupled equations (4), for the chosen parameters $k_1 = 1, k_2 = 2, \epsilon_1 = 1, \epsilon_2 = -1$ and suitable intervals for n and t.

Using the induction method like in [27] one can construct the N-soliton solution for the focusing case of semidiscrete cmKdV which has the form:

$$
\begin{aligned}
G_1 &= \sum_{\zeta=0,1} W_2(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i \eta_i + \sum_{1\le i<j}^{2N} \zeta_i \zeta_j \psi_{ij}\right) \\
G_2 &= \sum_{\zeta=0,1} W_2(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i [\eta_i + log(\epsilon_i)] + \sum_{1\le i<j}^{2N} \zeta_i \zeta_j \psi_{ij}\right) \\
F_1 &= \sum_{\zeta=0,1} W_1(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i \eta_i + \sum_{1\le i<j}^{2N} \zeta_i \zeta_j \psi_{ij}\right) \\
F_2 &= \sum_{\zeta=0,1} W_1(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i [\eta_i + log(\epsilon_i)] + \sum_{1\le i<j}^{2N} \zeta_i \zeta_j \psi_{ij}\right)
\end{aligned}
$$

where:

$$
\eta_j = k_j n + \omega_j t + \eta_j^{(0)}, \quad \eta_{j+N} = \eta_j^*, \quad k_{j+N} = k_j^*, \quad \log(\epsilon_{j+N}) = \log(\epsilon_j)^*, \quad \log(\epsilon_j) \in \{\pm\pi i\},
$$

$$
\omega_j = 2\epsilon_j \sinh(k_j), \quad \omega_{j+N} = \omega_j^*, \quad j = 1, ..., N,
$$

$$
e^{\psi_{ij}} = \begin{cases} \frac{1}{2}\left[\epsilon_i \epsilon_j \cosh(k_i + k_j) - 1\right]^{-1}, & \text{if } i = 1, ...N \text{ and } j = N+1, ..., 2N; \\ 2\left[\epsilon_i \epsilon_j \cosh(k_i - k_j) - 1\right], & \text{if } i = 1, ...N \text{ and } j = 1, ..., N \\ & \text{or } i = N+1, ..., 2N \text{ and } j = N+1, ..., 2N; \end{cases} \tag{9}
$$

$$
W_1(\underline{\zeta}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \zeta_i = \sum_{i=1}^{N} \zeta_{i+N} \\ 0, & \text{otherwise;} \end{cases}, \quad W_2(\underline{\zeta}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \zeta_i = 1 + \sum_{i=1}^{N} \zeta_{i+N} \\ 0, & \text{otherwise.} \end{cases}
$$

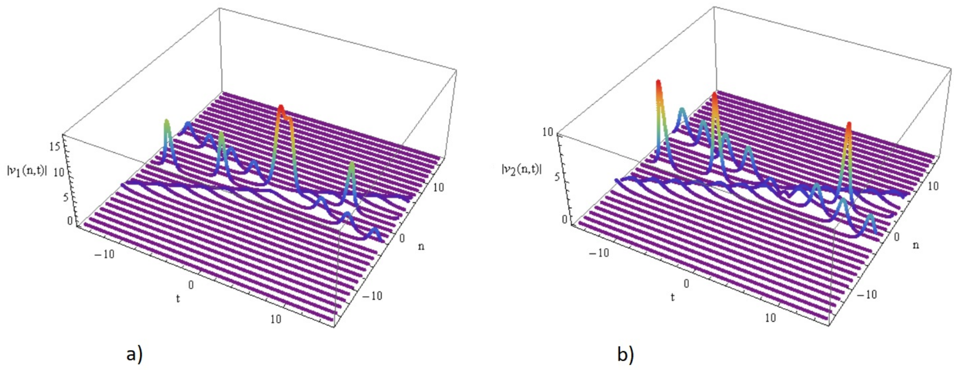

Figure 2. Plots of $|v_1(n,m)| = |G_1/F_1|$ (a) and $|v_2(n,m)| = |G_2/F_2|$ (b) given in (8), the absolute values of the 3-ss for the focusing semidiscrete cmKdV with two coupled equations (4), for the chosen parameters $k_1 = 1, k_2 = 2, k_3 = 3, \epsilon_1 = -1, \epsilon_2 = 1, \epsilon_3 = -1$ and suitable intervals for n and t.

4 Multi-soliton solutions for general semidiscrete cmKdV for $\forall M$

Starting from the general system of coupled semidiscrete cmKdV with any M equations (2) and using the nonlinear substitutions:

$$v_\mu = G_\mu / F_\mu, \quad \mu = 1, ..., M$$

we cast the general system in the Hirota bilinear form:

$$\begin{aligned} \mathbf{D_t} G_\mu \cdot F_\mu &= \overline{G_{\mu+1}} \underline{F_{\mu-1}} - \underline{G_{\mu-1}} \overline{F_{\mu+1}} \\ F_\mu^2 + |G_\mu|^2 &= \overline{F_{\mu+1}} \underline{F_{\mu-1}}. \end{aligned} \tag{10}$$

The N-soliton solution for the general system given in (2) has the following expressions for G_μ, F_μ, ($\mu = 1, ..., M$):

$$\begin{aligned} G_\mu &= \sum_{\zeta=0,1} W_2(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i [\eta_i + (\mu - 1)\log(\epsilon_i)] + \sum_{1 \le i < j}^{2N} \zeta_i \zeta_j \psi_{ij} \right), \\ F_\mu &= \sum_{\zeta=0,1} W_1(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i [\eta_i + (\mu - 1)\log(\epsilon_i)] + \sum_{1 \le i < j}^{2N} \zeta_i \zeta_j \psi_{ij} \right), \end{aligned} \tag{11}$$

where $W_1(\underline{\zeta})$, $W_2(\underline{\zeta})$, η_i, ψ_{ij} are given in (9) with the more complicated form of the dispersion relation due to the dependency on ϵ_j, which are now the roots of unity of order M:

$$\omega_j(k_j) = 2\left[\frac{\epsilon_j^2 + 1}{2\epsilon_j} \sinh k_j + \frac{\epsilon_j^2 - 1}{2\epsilon_j} \cosh k_j \right], \tag{12}$$

where:

$$\epsilon_j \in \left\{ e^{l \frac{2\pi i}{M}} \right\}, l = 1, ..M, j = 1, ..., N.$$

As expected, we have M possible branches of dispersion for each of the N solitons (k_j is the wave number of the j-soliton where $j = 1, ..., N$). The existence of multi-soliton solution proves the complete integrability of the investigated system.

5 The periodic reduction

An easier way of obtaining the same multi-soliton solution is offered by the periodic reduction technique which has as a starting point a general "diagonal" equation in two discrete dimensions. Considering that the independent discrete variable of semidiscrete cmKdV equation is a *diagonal* in a two-dimensional lattice, one can impose the periodic reduction on it and obtain coupled systems of semidiscrete cmKdV equations. To exemplify, let us start with the following completely integrable equation:

$$\frac{d}{dt}K_{n,m}(t) = (1 + |K_{n,m}(t)|^2)(K_{n+1,m+1}(t) - K_{n-1,m-1}(t)). \tag{13}$$

We consider the periodic 2-reduction on the m direction. Function $K(n,m)$ is periodic only with respect to m and the period is 2, which means that:

$$K(n,m) \equiv v_1(n),\ K(n,m+1) \equiv v_2(n),$$

$$K(n,m+2) \equiv v_1(n),\ K(n,m-1) \equiv v_2(n),$$

Introducing this reduction in (13) we get precisely system (4) analysed in Section 3:

$$\begin{aligned} \dot{v}_1 &= (1 + \alpha|v_1|^2)(\overline{v_2} - \underline{v_2}),\\ \dot{v}_2 &= (1 + \alpha|v_2|^2)(\overline{v_1} - \underline{v_1}). \end{aligned}$$

In the same way, if we impose periodic-3 reduction, we get system (5) with three equations and so on. Obviously, the general system of semidiscrete cmKdV (2) can be obtained from the diagonal semidiscrete cmKdV equation (13), choosing an M periodic reduction on the discrete variable m.

Considering $K_{n,m}(t) = G_n^m/F_n^m$, one can easily construct the Hirota bilinear form of (13):

$$\begin{aligned} \mathbf{D_t} G_n^m \cdot F_n^m &= G_{n+1}^{m+1} F_{n-1}^{m-1} - G_{n-1}^{m-1} F_{n+1}^{m+1}\\ (F_n^m)^2 + |G_n^m|^2 &= F_{n+1}^{m+1} F_{n-1}^{m-1}, \end{aligned} \tag{14}$$

where F_n^m is real function, while G_n^m is complex valued function (in this notation m is not exponent).

The 1-soliton solution is:

$$v_n^m = \frac{G_n^m}{F_n^m} = \frac{e^{\eta_1}}{1 + \frac{1}{2[\cosh(k_1+p_1+k_1^*+p_1^*)-1]} e^{\eta_1+\eta_1^*}},$$

$$\eta_1 = k_1 n + p_1 m + \omega_1 t + \eta_1^{(0)}, \quad \omega_1 = 2\sinh(k_1+p_1), \quad (\forall) \quad k_1, p_1 \in \mathbb{C}.$$

The explicit forms of the 2-soliton solutions:

$$G_n^m = e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+\eta_2^*+\psi_{12}+\psi_{14}+\psi_{24}} + e^{\eta_1+\eta_2+\eta_1^*+\psi_{12}+\psi_{13}+\psi_{23}}$$

$$F_n^m = 1 + e^{\eta_1+\eta_1^*+\psi_{13}} + e^{\eta_2+\eta_2^*+\psi_{24}} + e^{\eta_1+\eta_2^*+\psi_{14}} + e^{\eta_2+\eta_1^*+\psi_{23}} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+\sum\limits_{1\le i<j}^{4}\psi_{ij}}$$

where:

$$\eta_j = k_j n + p_j m + \omega_j t + \eta_j^{(0)}, \quad \omega_j = 2\sinh(k_j+p_j), \quad k_{j+2} = k_j^*, \quad p_{j+2} = p_j^*, \quad j = 1,2 \tag{15}$$

and

$$e^{\psi_{ij}} = \begin{cases} \frac{1}{2}\left[\cosh(k_i+p_i+k_j+p_j) - 1\right]^{-1}, & \text{if } i = 1,2 \text{ and } j = 3,4;\\ 2\left[\cosh(k_i+p_i-k_j-p_j) - 1\right], & \text{if } i = 1,2 \text{ and } j = 1,2\\ & \text{or } i = 3,4 \text{ and } j = 3,4. \end{cases}$$

The N-soliton solution has the following form for G_n^m and F_n^m:

$$G_n^m = \sum_{\underline{\zeta}=0,1} W_2(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i\eta_i + \sum_{1\leq i<j}^{2N} \zeta_i\zeta_j\psi_{ij}\right)$$

$$F_n^m = \sum_{\underline{\zeta}=0,1} W_1(\underline{\zeta}) \exp\left(\sum_{i=1}^{2N} \zeta_i\eta_i + \sum_{1\leq i<j}^{2N} \zeta_i\zeta_j\psi_{ij}\right)$$

where:

$$\eta_j = k_j n + p_j m + \omega_j t + \eta_j^{(0)}, \quad \eta_{j+N} = \eta_j^*, \quad j = 1, ..., N \tag{16}$$

$$\omega_j = 2\sinh(k_j + p_j), \quad \omega_{j+N} = \omega_j^*, \quad k_{j+N} = k_j^*, \quad p_{j+N} = p_j^*$$

$$e^{\psi_{ij}} = \begin{cases} \frac{1}{2}\left[\cosh(k_i + p_i + k_j + p_j) - 1\right]^{-1}, \\ \qquad \text{if } i = 1, ..., N \text{ and } j = N+1, ..., 2N; \\ 2\left[\cosh(k_i + p_i - k_j - p_j) - 1\right], \\ \qquad \text{if } i = 1, ..., N \text{ and } i = 1, ..., N \\ \qquad \text{or } i = N+1, ..., 2N \text{ and } j = N+1, ..., 2N; \end{cases}$$

$$W_1(\underline{\zeta}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \zeta_i = \sum_{i=1}^{N} \zeta_{i+N}; \\ 0, & \text{otherwise;} \end{cases}, \quad W_2(\underline{\zeta}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \zeta_i = 1 + \sum_{i=1}^{N} \zeta_{i+N}; \\ 0, & \text{otherwise.} \end{cases}$$

Comparing system (14) with (10), the bilinear forms for 2D semidiscrete cmKdV and general semidiscrete cmKdV, we reobtain the N-soliton solution for $M = 2$ or $\forall M$ constructed in the previous sections (3 and 4). The systems are obviously the same if we consider that the second index, m, of $K_{n,m}(t) = G_n^m/F_n^m$ in (14) becomes $\mu = \overline{1, M}$ in (10), parameter which indiciates the soliton solutions for the M-component semidiscrete cmKdV system.

For example, in the case $M = 2$, the m-dependence is dropped, p_j appearing in the definitions will be $-\pi$i , $+\pi$i, giving the dispersion relation two branches (allowing solitons to move either in the same direction or opposite to one another). For $M = 3$, again the m-dependence is dropped, p_j will have the values $-2\pi\mathrm{i}/3$, $+2\pi\mathrm{i}/3$ or 2πi (its exponentials are the cubic roots of the unity), we have three branches of the dispersion relation. For $\forall M$, dropping the m-dependence, $p_j \in \mu\frac{2\pi\mathrm{i}}{M}$, $\mu = \overline{1, M}$ (its exponentials are the M roots of unity), leading to M branches of dispersion given in (12). Considering this parallel, the periodic reduction proves again to be a very effective tool for deriving multi-solitons for multicomponent systems.

6 Conclusions

The paper discusses the focused coupled semidiscrete cmKdV equations with multiple branches of dispersion. The main motivation was to prove that the integrability survives in general coupled systems for $\forall M$ equations, to construct the multi-soliton solitons and offer a simple alternative method of direct construction of the multi-solitons using the periodic reduction. An interesting aspect in the structure of the dispersion relation, specific to this type of systems, is the existence of multiple branches of dispersion, and also the phases of the components that are parametrized by the order M roots of unity. Using the Hirota bilinear formalism, for $M = 2$ and for $\forall M$, we constructed the multi-solitons and proved complete integrability. The elastic interaction of solitons is obvious in the 2- and 3-soliton solution ploted for $M = 2$ for some chosen parameters. As already mentioned, applying periodic reduction, calculus leads to the same multi-soliton solutions for a multicomponent cmKdV system with any number of equations, starting from the corresponding 2D system. This simple method was initially applied in [17] and successfully used later in [18, 19] for coupled semidiscrete Volterra and Ablowitz-Ladik systems. In fact, for any completely integrable semidiscrete equation $F(\dot{u}_n, u_n, u_{n+1}, u_{n-1}) = 0$, constructing its 2D integrable variant $F(\dot{u}_{n,m} u_{n,m} u_{n+1,m+1}, u_{n-1,m-1}) = 0$, then by various periodic reduction on m, one can obtain integrable matrix systems.

References

[1] S. Tremblay, B. Grammaticos and A. Ramani, Phys. Lett. A **278**(6), 319 (2001).
[2] B. Grammaticos, A. Ramani and V. Papageorgiou, Phys. Rev. Lett. **67**, 1825 (1991).
[3] V. Adler, A. Bobenko and Y. Suris, Comm. Math. Phys. **233**(3), 513 (2003).
[4] P. D. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).
[5] N. C. Babalic, R. Constantinescu and V. S. Gerdjikov, Balk. J. Geom. Appl. **19**(2), 11 (2014).
[6] C.N. Babalic, R. Constantinescu and V.S. Gerdjikov, J. Geom. Symmetry Phys. **37**, 1-24 (2015).
[7] R. Cimpoiasu, V. Cimpoiasu and R. Constantinescu, Rom. J. Phys., **55** (1-2), 25-35 (2010).
[8] R. Cimpoiasu and A. S. Păuna, Open Physics **16**(1), 419 (2018).
[9] A. Babalean, R. Constantinescu and C. Ionescu, Journal of Physics A – Mathematical and General **32**(16), 3005 – 3012 (1999).
[10] R. Constantinescu and C. Ionescu, Cent. Eur. J. Phys. **7**(4), 711-720, (2009).
[11] R. Constantinescu and C. Ionescu, Ann. Phys. (Berl.) **15**(3), 169-176 (2006).
[12] R. Constantinescu and C. Ionescu, Int. J. Mod. Phys. A **21**(32), 6629-6640 (2006).
[13] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge Tracts in Mathematics 155, Cambridge University Press, Cambridge, 2004.
[14] N.C. Babalic, Romanian Journal of Physics **58**(5-6), 408-413, (2013).
[15] C.N. Babalic and A.S. Carstea, Theoretical and Mathematical Physics **188**(2) 1172-1180 (2016).
[16] C.N. Babalic and A.S. Carstea, Modern Physics Letters A **35**(17) (2020).
[17] A. S. Carstea, T. Tokihiro, J. Phys. A: Math. Theor. **48** (055205), (2015).
[18] C.N. Babalic, International Journal of Modern Physics B **34**(29) (2020).
[19] C.N. Babalic and A.S. Carstea, Journal of Physics A-Mathematical and Theoretical **50**(41) (2017).
[20] B.M. Herbst, M.J. Ablowitz and E. Ryan, Comput. Phys. Commun. **65**, 137-142 (1991).
[21] T.R. Taha, *A*dv. Comp. Meth. for Partial Diff. Eq., VII, edited by R. Vichnevetsky, IMACS, 1992, 721-725.
[22] T.R. Taha, Math. Comp. Sim. **37** 461-467 (1994).
[23] Li-Yuan Ma, Hai-Qiong Zhao, Shou-Feng Shen, Wen-Xiu Ma, Commun. Nonlin. Sci. Num. Sim. (2018), doi: 10.1016/j.cnsns.2018.07.037
[24] Hai-qiong Zhao and Guo-fu Yu, Chaos **27**, 043113 (2017); doi: 10.1063/1.4982204
[25] M. J. Ablowitz and J. F. Ladik, *Journal of Mathematical Physics* **17**, 1011 (1976).
[26] C.N. Babalic, *The soliton solutions for semidiscrete complex mKdV equation*, Book Series: ITM Web of Conferences, **34**, 03002 (2020).
[27] C.N. Babalic and A. S. Carstea, J. Phys. A: Math. Theor. **46**, 145205 (2013).