Fredholm property of regular hypoelliptic operators on the scales of multianisotropic spaces

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Abstract. This paper studies the Fredholm properties for a class of regular hypoelliptic operators. We establish necessary and sufficient conditions for a priori estimates for differential operators acting in multianisotropic Sobolev spaces in $\mathbb{R}^n$. Fredholm criteria and index invariance are obtained for a wide class of regular hypoelliptic operators on the special scales of multianisotropic weighted spaces.

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1 Introduction

The class of regular hypoelliptic operators is a special subclass of Hyormander's hypoelliptic operators which have very important applications (see [1]). They are a natural generalization of elliptic, parabolic, $2b$–parabolic and quasielliptic operators and were introduced in the late 60s-70s and studied by many authors: S. M. Nikolsky [2], V. P. Mikhailov [3], J. Friberg [4], L. R. Volevich, S. G. Gindikin. [5], H. G. Ghazaryan [6, 7] and others. The analysis of regular hypoelliptic operators has certain difficulties as corresponding characteristic polynomials are not homogeneous as in the elliptic case. Fredholm properties are studied for special classes of hypoelliptic operators in various functional spaces, but most of the results are related to elliptic and quasielliptic operators.

For elliptic operators the Fredholm property is studied on various scales of weighted spaces in $\mathbb{R}^n$ in the works of R. B. Lockhart, R. C. McOwen [8, 9], L. A. Bagirov [10], E. Schrohe [11] and many others. L. A. Bagirov [12], G. A. Karapetyan, A. A. Darbinyan [13] studied the Fredholm property of quiasielliptic operators in the special weighted anisotropic spaces in $\mathbb{R}^n$. Isomorphism properties are obtained in G. V. Demidenko's works (see [14, 15]) for quasielliptic operators with constant coefficients on the special scales of weighted spaces. A. A. Darbinyan and A .G. Tumanyan [16–18] studied a priori estimates and Fredholm criteria for quasielliptic operators with special variable coefficients in anisotropic weighted Sobolev spaces.

L. Rodino, P. Boggiatto, E. Buzano (see [19]) studied the Fredholm properties and the spectrum of special classes of pseudodifferential operators acting in multianisotropic spaces with special polynomial weights. In A. G. Tumanyan's work [20] Fredholm criteria are obtained for the special subclass of regular hypoelliptic operators.

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In this paper we obtain a priori estimates for a class of regular hypoelliptic operators, acting in multianisotropic Sobolev spaces in $\mathbb{R}^n$ (see Theorems 3.2 and 3.3). A regularizer is constructed for regular hypoelliptic operators with special variable coefficients (see Theorem 3.5). For the considered class of operators, we establish Fredholm criteria and index invariance on the special scale of multianisotropic spaces $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ (see Theorem 3.6). The considered scale of multianisotropic spaces is more general than in the previous works (see, for example, [18, 20]).

2 Basic notions and definitions

Definition 2.1 *A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called an n–normal operator, if the following conditions hold:*

1. *the image of operator A is closed* $\left(\mathrm{Im}(A) = \overline{\mathrm{Im}(A)}\right)$*;*
2. *the kernel of operator A is finite dimensional* $(\dim \mathrm{Ker}(A) < \infty)$.

An operator A is called a Fredholm operator if conditions 1-2 hold and

3. *the cokernel of operator A is finite dimensional*
$(\dim \mathrm{coker}(A) = \dim Y/\,\mathrm{Im}(A) < \infty)$.

The difference between the dimension of the kernel and the cokernel of operator A is called the index of the operator:

$$\mathrm{ind}\,(A) = \dim \mathrm{Ker}(A) - \dim \mathrm{coker}(A).$$

Definition 2.2 *For a bounded linear operator A, acting from a Banach space X to a Banach space Y, the bounded linear operators* $R_1 : Y \to X$ *and* $R_2 : Y \to X$ *are called respectively left and right regularizers if the following holds:* $R_1 A = I_X + T_1, A R_2 = I_Y + T_2$*, where* I_X, I_Y*– are the identity operators,* $T_1 : X \to X$ *and* $T_2 : Y \to Y$ *are compact operators. A bounded linear operator* $R : Y \to X$ *is called a regularizer for operator A, if it is a left and right regularizer.*

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean n-dimensional space, $\mathbb{Z}_+^n$, $\mathbb{N}^n$ be the sets of n-dimensional multi-indices and multi-indices with natural components respectively.

Definition 2.3 *Let* $\mathcal{N} \subset \mathbb{Z}_+^n$ *be a finite set of multi-indices,* $\mathcal{R} = \mathcal{R}(\mathcal{N})$ *be a minimum convex polyhedron containing all the points* $\mathcal{N}$*. A polyhedron* $\mathcal{R}$ *is called completely regular if the following holds: a)* $\mathcal{R}$ *is a complete polyhedron:* $\mathcal{R}$ *has a vertex at the origin and further vertices on each coordinate axes in* $\mathbb{R}^n$*; b) all components of the outer normals of* $(n-1)$*–dimensional non-coordinate faces of* $\mathcal{R}$ *are positive.*

Let $\mathcal{R}$ be a completely regular polyhedron. Denote by $\mathcal{R}_j^{n-1}$ $(j = 1, \ldots, I_{n-1})$ $(n-1)$–dimensional non-coordinate faces of $\mathcal{R}$ with the corresponding outer normals μ^j such that all multi-indices $\alpha \in \mathcal{R}_j^{n-1}$ satisfy $(\alpha : \mu^j) = \frac{\alpha_1}{\mu_1^j} + ... + \frac{\alpha_n}{\mu_n^j} = 1$, $\partial\mathcal{R} = \bigcup_{j=1}^{I_{n-1}} \mathcal{R}_j^{n-1}$. For $k > 0$ denote by $k\mathcal{R} := \{k\alpha = (k\alpha_1, k\alpha_2 \ldots, k\alpha_n) : \alpha \in \mathcal{R}\}$.

Consider the differential form

$$P\,(x, \mathbb{D}) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) D^\alpha, \tag{1}$$

where $D^{\alpha} = D_1^{\alpha_1} ... D_n^{\alpha_n}$, $D_j = i^{-1} \frac{\partial}{\partial x_j}$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $a_\alpha(x) \in C(\mathbb{R}^n)$.

Denote by

$$P(x, \xi) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) \xi^\alpha. \tag{2}$$

For $\xi \in \mathbb{R}^n$ denote by

$$|\xi|_{\mathcal{R}} = \sum_{\alpha \in \mathcal{R}} |\xi^\alpha|, |\xi|_{\partial\mathcal{R}} = \sum_{\alpha \in \partial\mathcal{R}} |\xi^\alpha|.$$

Definition 2.4 *A differential form $P(x, \mathbb{D})$ is called regular at a point $x_0 \in \mathbb{R}^n$, if there exists a constant $\delta > 0$ such that:*

$$1 + |P(x_0, \xi)| \geq \delta |\xi|_{\mathcal{R}}, \forall \xi \in \mathbb{R}^n.$$

$P(x, \mathbb{D})$ is called regular in $\mathbb{R}^n$, if $P(x, \mathbb{D})$ is regular at each point $x \in \mathbb{R}^n$.

$P(x, \mathbb{D})$ is called uniformly regular in $\mathbb{R}^n$, if there exists a constant $\delta > 0$ such that:

$$1 + |P(x, \xi)| \geq \delta |\xi|_{\mathcal{R}}, \forall \xi \in \mathbb{R}^n, \forall x \in \mathbb{R}^n.$$

Example 2.1 *Examples of regular differential forms:*

1. *Let $m \in \mathbb{N}$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0, 0, \ldots, 0)$, $(m, 0, \ldots, 0), \ldots, (0, 0, \ldots, m)$. In this case conditions from definition 2.4 coincide with ellipticity conditions.*

2. *Let $\nu \in \mathbb{N}^n$ and $\mathcal{R}$ be a Newton polyhedron for the set of points $(0, 0, \ldots, 0)$, $(\nu_1, 0, \ldots, 0), \ldots, (0, 0, \ldots, \nu_n)$. In this case conditions from definition 2.4 coincide with quasiellipticity conditions.*

3. *Let $n = 2$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0, 0)$, $(8, 0)$, $(0, 8)$ and $(6, 4)$. Then*
$$P(x, \mathbb{D}) = a_1 D_1^8 + a_2 D_1^6 D_2^4 + a_3 D_2^8 + q(x)$$
is a regular differential form in $\mathbb{R}^2$ with some $a_1, a_2, a_3 > 0$ and $q \in C(\mathbb{R}^2)$.

4. *Let $n = 3$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0, 0, 0)$, $(8, 0, 0)$, $(0, 8, 0)$, $(6, 4, 0)$, $(6, 0, 6)$, $(0, 6, 6)$ and $(0, 0, 12)$. Then*
$$P(x, \mathbb{D}) = D_1^8 + D_1^6 D_2^4 + D_2^8 + D_1^6 D_3^6 + D_2^6 D_3^6 + D_3^{12} + q(x)$$
is a regular differential form in $\mathbb{R}^3$ with $q \in C(\mathbb{R}^3)$.

Denote

$$Q := \{g \in C(\mathbb{R}^n) : \exists c > 0 \text{ such that } g(x) \geq c > 0, \forall x \in \mathbb{R}^n\}.$$

For $m \in \mathbb{Z}_+$ and a completely regular polyhedron $\mathcal{R}$ denote

$$Q^{m,\mathcal{R}} := \Bigg\{ g \in Q : D^\beta g(x) \in C(\mathbb{R}^n), \frac{1}{g(x)} \rightrightarrows 0, \\ \max_{|x-y| \leq 1} \frac{|g(x) - g(y)|}{g(y)} \rightrightarrows 0 \text{ as } |x| \to \infty, \exists C_\beta > 0 \text{ s.t. } \frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\mu^j)}} \leq C_\beta, \forall x \in \mathbb{R}^n \\ \forall \beta \in m\mathcal{R}, \beta \neq 0, j = 1, \ldots, I_{n-1} \Bigg\}.$$

The considered class $Q^{m,\mathcal{R}}$ includes polynomial functions and special exponential functions such as:

$$(1+|x|_{\mathcal{R}})^l, l>0, \exp(1+|x|_{\mathcal{R}})^r, 0<r<\frac{1}{\mu_{max}},$$

where $\mu_{max} = \max\limits_{1\le i\le I_{n-1}} \max\limits_{1\le s\le n} \left\{\mu_s^i\right\}$.

For $k\in\mathbb{R}$, a completely regular polyhedron $\mathcal{R}$ and $1<p<\infty$ denote

$$H^{k,\mathcal{R},p}(\mathbb{R}^n) := \left\{u\in S' : \|u\|_{k,\mathcal{R},p} := \|F^{-1}(1+|\xi|_{\partial\mathcal{R}})^k Fu\|_{L_p(\mathbb{R}^n)} < \infty\right\},$$

where S' is the set of tempered distributions.

For $k\in\mathbb{Z}_+$, a completely regular polyhedron $\mathcal{R}$, $1<p<\infty$ and $q\in Q$ denote

$$H_q^{k,\mathcal{R},p}(\mathbb{R}^n) := \left\{u : \|u\|_{H_q^{k,\mathcal{R},p}(\mathbb{R}^n)} := \|u\|_{k,\mathcal{R},p,q} := \sum_{\alpha\in k\mathcal{R}} \left\|D^\alpha u\cdot q^{k-\max\limits_i(\alpha:\mu^i)}\right\|_{L_p(\mathbb{R}^n)} < \infty\right\}.$$

For $\Omega\subset\mathbb{R}^n$ and $x_0\in\mathbb{R}^n$ denote

$$H_q^{k,\mathcal{R},p}(\Omega) := \left\{u : \|u\|_{H_q^{k,\mathcal{R},p}(\Omega)} := \sum_{\alpha\in k\mathcal{R}} \left\|D^\alpha u\cdot q^{k-\max\limits_i(\alpha:\mu^i)}\right\|_{L_p(\Omega)} < \infty\right\},$$

$$H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n) := \left\{u : \|u\|_{H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n)} := \|u\|_{k,\mathcal{R},p,q(x_0)} := \sum_{\alpha\in k\mathcal{R}} \left\|D^\alpha u\cdot q(x_0)^{k-\max\limits_i(\alpha:\mu^i)}\right\|_{L_p(\mathbb{R}^n)} < \infty\right\}.$$

For $k\in\mathbb{Z}_+$ it can be checked that with some constants $C_1, C_2>0$ we have:

$$C_1\|F^{-1}(q(x_0)+|\xi|_{\partial\mathcal{R}})^k Fu\|_{L_p(\mathbb{R}^n)} \le \|u\|_{k,\mathcal{R},p,q(x_0)} \le C_2\|F^{-1}(q(x_0)+|\xi|_{\partial\mathcal{R}})^k Fu\|_{L_p(\mathbb{R}^n)}, \quad \forall u\in H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n). \tag{3}$$

3 Main results

Let $k\in\mathbb{Z}_+$ and $q\in Q$. Consider the differential form $P(x,\mathbb{D})$ (see (1)) with the coefficients that satisfy the following conditions:

$$P(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x)D^\alpha = \sum_{\alpha\in\mathcal{R}}\left(a_\alpha^0(x)q(x)^{1-\max\limits_i(\alpha:\mu^i)} + a_\alpha^1(x)\right)D^\alpha, \tag{4}$$

where $a_\alpha(x) = a_\alpha^0(x)q(x)^{1-\max\limits_i(\alpha:\mu^i)} + a_\alpha^1(x)$, $D^\beta(a_\alpha^0(x)) = O\left(q(x)^{\min\limits_i(\beta:\mu^i)}\right)$ and $D^\beta(a_\alpha^1(x)) = o\left(q(x)^{1-\max\limits_i(\alpha-\beta:\mu^i)}\right)$ when $|x|\to\infty$ for all $\alpha\in\mathcal{R},\beta\in k\mathcal{R}$.

It is easy to check that $P(x,\mathbb{D})$ generates a bounded linear operator, acting from $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$.

Denote

$$P_0(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} a_\alpha^0(x)D^\alpha, \tag{5}$$

$$L(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} a^1_\alpha(x) D^\alpha. \tag{6}$$

For $N > 0$ and $x_0 \in \mathbb{R}^n$ denote

$$K_N(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \le N\}, K_N := K_N(0).$$

Consider the unity partition from the work [20]. Let $0 < \delta_0 < 1$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^n$ and $\varphi(x) = 1$ for $x \in K_{\delta_0/2}$, $\varphi(x) = 0$ for $|x| \ge \delta_0$ and $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset K_{2\delta_0}$ and $\psi(x) = 1$ for $x \in K_{\delta_0}$. Let $\omega > 0$ be such that $\omega\sqrt{n} < \delta_0$. Denote $\{z_m\}_{m=0}^\infty$ points on the lattice in $\mathbb{R}^n$ with a side equal to ω.

Denote

$$\varphi_m(x) := \varphi(x - z_m)\left(\sum_{l=0}^\infty \varphi(x - z_l)\right)^{-1}, \psi_m(x) := \psi(x - z_m), \quad m \in 0, 1, \dots. \tag{7}$$

Denote $W_m = \operatorname{supp}\varphi_m, m = 0, 1, 2, \dots$.

Then $\{\varphi_m\}_{m=0}^\infty$ and $\{\psi_m\}_{m=0}^\infty$ satisfy the following conditions:

(i). $\max\limits_{x,y\in\operatorname{supp}\varphi_m} |x - y| < \delta_0$,

(ii). there exists $r \in \mathbb{N}$ such that for any number i there are no more than r functions $\varphi_j(x)$ such that $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j \ne \varnothing$;

(iii). $\varphi_m(x)\psi_m(x) \equiv \varphi_m(x)$ for all $x \in \mathbb{R}^n$;

(iv). for any $\alpha \in \mathbb{Z}_+^n$ there exists a constant $C_\alpha > 0$ such that $|D^\alpha\varphi_m(x)| \le C_\alpha$, $|D^\alpha\psi_m(x)| \le C_\alpha$, $\forall x \in \mathbb{R}^n, m = 0, 1, \dots$;

(v). $\sum\limits_{m=0}^\infty \varphi_m(x) \equiv 1$.

3.1 A priori estimates in multianisotropic spaces

Further we use the following theorem which is a consequence of Theorem 7.1 from [22] for the spaces $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$:

Theorem 3.1 *Let $k \in \mathbb{Z}_+, q \in Q$ and $P(x,\mathbb{D})$ be the differential form* (4)*. Then operator $P(x,\mathbb{D})$, acting from $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ is an $n-$normal operator if and only if there exist $\kappa > 0$ and $N > 0$ such that*

$$||u||_{k+1,\mathcal{R},p,q} \le \kappa\left(||Pu||_{k,\mathcal{R},p,q} + ||u||_{L_p(K_N)}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \tag{8}$$

Theorem 3.2 *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be the differential form* (4) *with the coefficients that satisfy $\lim\limits_{|x|\to\infty}\max\limits_{|x-y|\le 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$.*

Let there exist a constant $\kappa > 0$ such that:

$$||u||_{k+1,\mathcal{R},p,q} \le \kappa\left(||Pu||_{k,\mathcal{R},p,q} + ||u||_{L_p(\mathbb{R}^n)}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \tag{9}$$

Then $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\lambda^{1-\max\limits_i(\alpha:\mu^i)}\xi^\alpha\right| \ge \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall\xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \tag{10}$$

Proof. The proofs of Theorem 3.1 and Theorem 3.3 from [20] for $p = 2$ can be generalized for the spaces $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ in a similar way. □

Further we establish that the regularity of $P(x, \mathbb{D})$ in $\mathbb{R}^n$ and condition (10) are not only the necessary conditions but also sufficient for the fulfillment of the a priori estimate (9) in spaces $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$.

Lemma 3.1 *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}, x_0 \in \mathbb{R}^n$ and $P_0(x, \mathbb{D})$ be the differential form* (5). *Let there exist a constant $\delta > 0$ such that*

$$|P_0(x_0, \xi)| \geq \delta(q(x_0) + |\xi|_{\partial\mathcal{R}}), \forall\xi \in \mathbb{R}^n. \tag{11}$$

Then operator $P_0(x_0, \mathbb{D})$, acting from $H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ to $H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n)$, has a bounded linear inverse operator R_0, acting from $H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n)$ to $H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n)$.

Proof. Let $u \in H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ and $P_0(x_0, \mathbb{D})u = f$. First, we prove that $u \in H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ is the unique solution of $P_0(x_0, \mathbb{D})u = f$. Applying the Fourier transform, we obtain

$$P_0(x_0, \xi)Fu = Ff.$$

Hence

$$Fu = \frac{Ff}{P_0(x_0, \xi)}.$$

$$(q(x_0) + |\xi|_{\partial\mathcal{R}})^{k+1} Fu = (q(x_0) + |\xi|_{\partial\mathcal{R}})^k \frac{q(x_0) + |\xi|_{\partial\mathcal{R}}}{P_0(x_0, \xi)} Ff.$$

Since (11) is satisfied it is easy to check that $\frac{q(x_0)+|\xi|_{\partial\mathcal{R}}}{P_0(x_0,\xi)}$ is a Fourier multiplier (see [6]).
Using this fact and the norm equivalence (3), with some constants $C_1, C_2, C_3 > 0$ we get

$$\begin{aligned} \|u\|_{k+1,\mathcal{R},p,q(x_0)} &\leq C_1 \left\|F^{-1} (q(x_0) + |\xi|_{\partial\mathcal{R}})^{k+1} Fu\right\|_{L_p(\mathbb{R}^n)} \\ &= C_1 \left\|F^{-1} \frac{q(x_0) + |\xi|_{\partial\mathcal{R}}}{P_0(x_0, \xi)} F\left(F^{-1} (q(x_0) + |\xi|_{\partial\mathcal{R}})^k Ff\right)\right\|_{L_p(\mathbb{R}^n)} \\ &\leq C_2 \left\|F^{-1} (q(x_0) + |\xi|_{\partial\mathcal{R}})^k Ff\right\|_{L_p(\mathbb{R}^n)} \leq C_3\|f\|_{k,\mathcal{R},p,q(x_0)}. \end{aligned} \tag{12}$$

We have

$$\|u\|_{k+1,\mathcal{R},p,q(x_0)} \leq C_3\|Pu\|_{k,\mathcal{R},p,q(x_0)}, \forall u \in H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n). \tag{13}$$

Thus, the uniqueness is proved.

Let $f \in H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n)$. Consider $u = F^{-1}\left(\frac{1}{P_0(x_0,\xi)}Ff\right)$. Similarly, it can be shown that $u \in H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ and (13) holds.

So, we get that for the operator $P_0(x_0, \mathbb{D}) : H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n)$ there exists a bounded linear inverse operator $R_0 : H_{q(x_0)}^{k,\mathcal{R},p}(\mathbb{R}^n) \to H_{q(x_0)}^{k+1,\mathcal{R},p}(\mathbb{R}^n)$. □

Theorem 3.3 *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x, \mathbb{D})$ be the differential form* (4) *with the coefficients that satisfy $\lim\limits_{|x|\to\infty} \max\limits_{|x-y|\leq 1} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathcal{R}$. Let $P(x, \mathbb{D})$ be regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that*

$$\left|\sum_{\alpha\in\mathcal{R}} a_\alpha^0(x)\lambda^{1-\max\limits_i(\alpha:\mu^i)}\xi^\alpha\right| \geq \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall\xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \tag{14}$$

Then there exist constants $\kappa > 0$ and $N > 0$ such that:

$$\|u\|_{k+1,\mathcal{R},p,q} \leq \kappa\left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_N)}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \tag{15}$$

Proof. Let $m_0 \in \mathbb{N}$. Using the properties of functions $\{\varphi_m\}_{m=0}^{\infty}$ it is easy to check that with some $C > 0$ the following holds

$$\|u\|^p_{k+1,\mathcal{R},p,q} \le C\left(\sum_{m=0}^{m_0} \|\varphi_m u\|^p_{k+1,\mathcal{R},p,q} + \sum_{m=m_0+1}^{\infty} \|\varphi_m u\|^p_{k+1,\mathcal{R},p,q}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \quad (16)$$

Using the a priori estimates for bounded domains from the work [23] with some constants $C_1 > 0$ and $N_1 > 0$ we have

$$\sum_{m=1}^{m_0} \|\varphi_m u\|_{k+1,\mathcal{R},p,q} \le C_1\left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n), \quad (17)$$

where N_1 is such that $\bigcup_{i=0}^{m_0} W_i \subset K_{N_1}$.

Denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} \left[\psi_m(x)\left(a^0_\alpha(x) q(x)^{1-\max_i(\alpha:\mu^i)} - a^0_\alpha(x_m) q(x_m)^{1-\max_i(\alpha:\mu^i)}\right)\right.$$
$$\left. + a^0_\alpha(x_m) q(x_m)^{1-\max_i(\alpha:\mu^i)}\right] D^\alpha, m = 0, 1, \ldots.$$

Using the properties of functions $\{\psi_m\}_{m=0}^{\infty}$, the condition on the coefficients $\lim_{|x|\to\infty} \max_{|x-y|\le 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$ and the weight function $q \in Q^{k,\mathcal{R}}$ it can be shown that for all $\alpha \in \mathcal{R}$ and $\beta \in k\mathcal{R}$ and $m > m_0$ the following holds

$$\left|D^\beta\left(\psi_m(x)\left(a^0_\alpha(x) q(x)^{1-\max_i(\alpha:\mu^i)} - a^0_\alpha(x_m) q(x_m)^{1-\max_i(\alpha:\mu^i)}\right)\right)\right|$$
$$\le \tau_{\alpha,\beta}(m_0) q(x_m)^{1-max_i(\alpha-\beta:\mu^j)},$$

where $\tau_{\alpha,\beta}(m_0) \to 0$ when $m_0 \to \infty$.

Using the last inequality and Lemma 3.1, similarly to the proof of Theorem 2.2 from the work [13], it is easy to check that for a big enough m_0 operators $P^m(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ for $m > m_0$ have bounded inverse operators. Since (10) holds they have uniformly bounded norms and with some $C_2 > 0$ the following holds

$$\|\varphi_m u\|_{k+1,\mathcal{R},p,q} \le C_2 \|P^m(\varphi_m u)\|_{k,\mathcal{R},p,q}, \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

Since $P^m(\varphi_m u) = P_0(\varphi_m u)$ for all $u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ and $m = m_0+1, m_0+2, \ldots$ we get

$$\|\varphi_m u\|_{k+1,\mathcal{R},p,q} \le C_2 \|P^m(\varphi_m u)\|_{k,\mathcal{R},p,q} \le C_2 \|P_0(\varphi_m u)\|_{k,\mathcal{R},p,q}, \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

Using the properties of functions $\{\varphi_m\}_{m=0}^{\infty}$ and the weight function $q \in Q^{k,\mathcal{R}}$ it can be shown that if m_0 is big enough for $m > m_0$ with some constants $C_3, C_4 > 0$ we have

$$\|\varphi_m P_0 u - P_0(\varphi_m u)\|^p_{k,\mathcal{R},p,q}$$
$$\le C_3 \left\|\sum_{\alpha\in\mathcal{R}} \sum_{\beta+\gamma=\alpha, |\gamma|>0} a^0_\alpha(x) D^\beta u D^\gamma \varphi_m q(x)^{1-\max_i(\alpha:\mu^i)}\right\|^p_{k,\mathcal{R},p,q}$$
$$\le C_4 \left\|\sum_{\alpha\in\mathcal{R}} \sum_{\beta+\gamma=\alpha, |\gamma|>0} a^0_\alpha(x) D^\beta u D^\gamma \varphi_m \frac{1}{q(x)^{\min_i(\gamma:\mu^i)}} q(x)^{1-\max_i(\beta:\mu^i)}\right\|^p_{k,\mathcal{R},p,q}$$
$$\le \omega_1(m_0) \|u\|^p_{H_q^{k+1,\mathcal{R},p}(W_m)},$$

where $\omega_1(m_0) \to 0$ when $m_0 \to \infty$.

From the last two inequalities with some constant $C_5 > 0$ we get

$$\|\varphi_m u\|^p_{k+1,\mathcal{R},p,q} \leq C_5 \left(\|\varphi_m P_0 u\|^p_{k,\mathcal{R},p,q} + \omega_1(m_0)\|u\|^p_{H_q^{k+1,\mathcal{R},p}(W_m)} \right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

Summing up for all $m > m_0$ and taking into account the property (ii) of $\{\varphi_m\}_{m=0}^\infty$ and $\{W_m\}_{m=0}^\infty$, with some constant $C_6 > 0$ we get

$$\sum_{m=m_0+1}^{\infty} \|\varphi_m u\|^p_{k+1,\mathcal{R},p,q} \leq C_6 \left(\|P_0 u\|^p_{k,\mathcal{R},p,q} + \omega_1(m_0)\|u\|^p_{k+1,\mathcal{R},p,q} \right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \tag{18}$$

From (16), (17) and (18) we get that with some constant $C_7 > 0$

$$\|u\|_{k+1,\mathcal{R},p,q} \leq C_7 \left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} + \|P_0 u\|_{k,\mathcal{R},p,q} + \omega_2(m_0)\|u\|_{k+1,\mathcal{R},p,q} \right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n), \tag{19}$$

where $\omega_2(m_0) \to 0$ when $m_0 \to \infty$.

Since $P_0(x, \mathbb{D}) = P(x, \mathbb{D}) - L(x, \mathbb{D})$ we have

$$\|P_0 u\|_{k,\mathcal{R},p,q} \leq \|Pu\|_{k,\mathcal{R},p,q} + \|Lu\|_{k,\mathcal{R},p,q}, \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

Taking into account the conditions $D^\beta(a^1_\alpha(x)) = o\left(q(x)^{1-\max_i(\alpha-\beta:\mu^i)}\right)$ when $|x| \to \infty$, $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$ it is easy to check that

$$\|Lu\|_{k,\mathcal{R},p,q} \leq \omega_3(m_0)\|u\|_{k+1,\mathcal{R},p,q} + C_8\|u\|_{H^{k+1,\mathcal{R},p}(K_{N_1})}, \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n),$$

where $\omega_3(m_0) \to 0$ when $m_0 \to \infty$ and N_1 is such that $\bigcup_{i=0}^{m_0} W_i \subset K_{N_1}$.

Similarly to (17), using the a priori estimate from [23], with some constant $C_9 > 0$ we get

$$\|Lu\|_{k,\mathcal{R},p,q} \leq \omega_3(m_0)\|u\|_{k+1,\mathcal{R},p,q} + C_9 \left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} \right).$$

From the last estimate and (19) we get

$$\begin{aligned}
\|u\|_{k+1,\mathcal{R},p,q} &\leq C_7 \left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} + \|P_0 u\|_{k,\mathcal{R},p,q} + \omega_2(m_0)\|u\|_{k+1,\mathcal{R},p,q} \right) \\
&\leq C_7 \left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} + \omega_3(m_0)\|u\|_{k+1,\mathcal{R},p,q} + C_9 \left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} \right) \right. \\
&\quad \left. + \omega_2(m_0)\|u\|_{k+1,\mathcal{R},p,q} \right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).
\end{aligned}$$

We can take m_0 big enough such that $C_7 (\omega_3(m_0) + \omega_2(m_0)) < 1/2$.

Then, with some constant $C_{10} > 0$ we get

$$\|u\|_{k+1,\mathcal{R},p,q} \leq C_{10} \left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_{N_1})} \right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

□

Corollary 3.1 *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x, \mathbb{D})$ be the differential form* (4) *with the coefficients that satisfy $\lim_{|x|\to\infty} \max_{|x-y|\leq 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$. Let $P(x, \mathbb{D})$ be regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that*

$$\left| \sum_{\alpha\in\mathcal{R}} a^0_\alpha(x) \lambda^{1-\max_i(\alpha:\mu^i)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \tag{20}$$

Then operator $P(x, \mathbb{D})$, acting from $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$, is an $n-$normal operator.

Proof. Applying Theorem 3.3, we get the a priori estimate (8). Thus, from Theorem 3.1 it follows that $P(x, \mathbb{D})$, acting from $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$, is an $n-$normal operator. □

3.2 Fredholm criteria for a class of regular hypoelliptic operators

Further we use the following theorem (see Theorem 3.14 [21]):

Theorem 3.4 *Let A be a bounded linear operator acting from a Banach space X to a Banach space Y. Then the following holds:*

1. *if operator A has a left regularizer, then kernel of operator A in X is finite dimensional;*
2. *if operator A has a right regularizer, then the image of operator A is closed in Y and the cokernel is finite dimensional;*
3. *operator A has left and right regularizers if and only if A is a Fredholm operator.*

Theorem 3.5 *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be the differential form* (4) *with the coefficients that satisfy $\lim\limits_{|x|\to\infty} \max\limits_{|x-y|\le 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$.*

Then operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ is a Fredholm operator if and only if $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\lambda^{1-\max\limits_i(\alpha:\mu^i)}\xi^\alpha\right| \ge \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall\xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \tag{21}$$

Proof. Let $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ be a Fredholm operator. Since Fredholm operator is $n-$ normal, then from Theorem 3.1 the a priori estimate (8) is satisfied. From Theorem 3.2 we get that $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and (21) holds. So, the necessary part is proved.

Consider the unity partition (7). Let $m_0 \in \mathbb{N}$ and $x_m \in W_m, m = 0, 1, 2 \ldots$.

For $m \le m_0$ denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} (\psi_m(x)(a_\alpha(x) - a_\alpha(x_m)) + a_\alpha(x_m))D^\alpha,$$

$$P^{m,0}(x,\mathbb{D}) := \sum_{\alpha\in\partial\mathcal{R}} (\psi_m(x)(a_\alpha(x) - a_\alpha(x_m)) + a_\alpha(x_m))D^\alpha,$$

$$R^{m,0} := F^{-1}\frac{|\xi|_{\partial\mathcal{R}}}{(1 + |\xi|_{\partial\mathcal{R}})P^{m,0}(x_m,\xi)}F.$$

Since $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, applying Lemma 2.1 from [20], we get that if δ_0 from definitions of (7) is small enough the following representation holds for $m \le m_0$:

$$P^m(x,\mathbb{D})R^{m,0} = I + T^m_1 + T^m_2, \tag{22}$$

where $T^m_1 : H^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H^{k+1+\sigma,\mathcal{R},p}(\mathbb{R}^n)$ with $\sigma = \sigma(\mathcal{R}) > 0$ and operator $T^m_2 : H^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ satisfies $||T^m_2|| < 1$. Denote

$$R^m := R^{m,0}(I + T^m_2)^{-1}.$$

Then the following holds:

$$P^mR^m = I + T^m, \tag{23}$$

where $T^m : H^{k,\mathcal{R},p}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R},p}(\mathbb{R}^n)$ with some $\sigma = \sigma(\mathcal{R}) > 0$.

For $m > m_0$ denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} \left[\psi_m(x)\left(a^0_\alpha(x)q(x)^{1-\max_i(\alpha:\mu^i)} - a^0_\alpha(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}\right) + a^0_\alpha(x_m)q(x_m)^{1-\max_i(\alpha:\mu^i)}\right] D^\alpha.$$

Similarly as in the proof of Theorem 3.3 we can take m_0 big enough such that for $m > m_0$ operators $P^m : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ have uniformly bounded inverse operators $R^m : H^{k,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$.

Consider

$$Rf := \sum_{l=0}^{\infty} \psi_l R^l(\varphi_l f),\ f \in H^{k,\mathcal{R},p}_q(\mathbb{R}^n). \tag{24}$$

Taking into account that the norms of operators $R^l : H^{k,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$ are uniformly bounded, $q \in Q^{k,\mathcal{R}}$ and the properties of functions $\{\varphi_m\}_{m=1}^\infty, \{\psi_m\}_{m=1}^\infty$, it is easy to check that R is a bounded linear operator, acting from $H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ to $H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$. Similarly to the proof of Theorem 2.6 from the work [20] it can be checked that $R : H^{k,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)$ is a right and left regularizer. Then, from Theorem 3.4 we get that operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ has a finite dimensional cokernel. Applying Corollary 3.1, we get that operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ is n–normal. So, operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ is a Fredholm operator. □

Denote

$$\operatorname{Ker}\left(P; H^{k,\mathcal{R},p}_q\right) := \left\{u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) : P(x,\mathbb{D})u = 0\right\},$$

$$\operatorname{Im}\left(P; H^{k,\mathcal{R},p}_q\right) := \left\{f \in H^{k,\mathcal{R},p}_q(\mathbb{R}^n) : \exists u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n)\ \ P(x,\mathbb{D})u = f\right\},$$

$$\operatorname{coker}\left(P; H^{k,\mathcal{R},p}_q\right) := H^{k,\mathcal{R},p}_q(\mathbb{R}^n) \Big/ \overline{\operatorname{Im}\left(P; H^{k,\mathcal{R},p}_q\right)},$$

$$\operatorname{ind}\left(P; H^{k,\mathcal{R},p}_q\right) := \dim\operatorname{Ker}\left(P; H^{k,\mathcal{R},p}_q\right) - \dim\operatorname{coker}\left(P; H^{k,\mathcal{R},p}_q\right).$$

Corollary 3.2 *Let $k \in \mathbb{N}, q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be the differential form* (4) *with the coefficients that satisfy $\lim_{|x|\to\infty} \max_{|x-y|\le 1} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$.*

Then for $k_1, k_2 \in \mathbb{N}$ such that $0 \le k_1 < k_2 \le k$ operators $P(x,\mathbb{D}) : H^{k_1+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k_1,\mathcal{R},p}_q(\mathbb{R}^n)$ and $P(x,\mathbb{D}) : H^{k_2+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k_2,\mathcal{R},p}_q(\mathbb{R}^n)$ are Fredholm operators and

$$\dim\operatorname{Ker}(P; H^{k_1,\mathcal{R},p}_q) = \dim\operatorname{Ker}(P; H^{k_2,\mathcal{R},p}_q),$$

$$\dim\operatorname{coker}(P; H^{k_1,\mathcal{R},p}_q) = \dim\operatorname{coker}(P; H^{k_2,\mathcal{R},p}_q),$$

$$\operatorname{ind}(P; H^{k_1,\mathcal{R},p}_q) = \operatorname{ind}(P; H^{k_2,\mathcal{R},p}_q).$$

Proof. From Theorem 3.5 we get that $P(x,\mathbb{D}) : H^{k_1+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k_1,\mathcal{R},p}_q(\mathbb{R}^n)$ and $P(x,\mathbb{D}) : H^{k_2+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k_2,\mathcal{R},p}_q(\mathbb{R}^n)$ are Fredholm operators which have the regularizer (24). Then for all $u \in H^{k_1+1,\mathcal{R},p}_q(\mathbb{R}^n)$ we have

$$RP(x,\mathbb{D})u = u + Tu, \tag{25}$$

where $T : H^{k_1+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k_1+1,\mathcal{R},p}_q(\mathbb{R}^n)$ is a compact operator.

Since $H_q^{k_2+1,\mathcal{R},p}(\mathbb{R}^n)$ is dense in $H_q^{k_1+1,\mathcal{R},p}(\mathbb{R}^n)$ when $k_2 > k_1$ and T is a compact operator, applying Theorem 8.3.9 from [24], we get the following representation:

$$T = T_1 + T_2,$$

where $T_1 : H_q^{k_1+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k_1+1,\mathcal{R},p}(\mathbb{R}^n)$, such that $\|T_1\| < 1$ and $T_2 : H_q^{k_1+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k_2+1,\mathcal{R},p}(\mathbb{R}^n)$.

Applying $(I + T_1)^{-1}$ to the both sides of (25) , we get

$$(I + T_1)^{-1} RPu = u + (I + T_1)^{-1} T_2 u.$$

Let $u \in \operatorname{Ker}\left(P; H_q^{k_1,\mathcal{R},p}\right)$. Then $u = -(I + T_1)^{-1} T_2 u \in H_q^{k_2+1,\mathcal{R},p}(\mathbb{R}^n)$.

So, it is proved that $\operatorname{Ker}\left(P; H_q^{k_1,\mathcal{R},p}\right) \subset \operatorname{Ker}\left(P; H_q^{k_2,\mathcal{R},p}\right)$. Since $k_1 < k_2$ we also have $\operatorname{Ker}\left(P; H_q^{k_2,\mathcal{R},p}\right) \subset \operatorname{Ker}\left(P; H_q^{k_1,\mathcal{R},p}\right)$. Thus, we obtain

$$\dim \operatorname{Ker}\left(P; H_q^{k_1,\mathcal{R},p}\right) = \dim \operatorname{Ker}\left(P; H_q^{k_2,\mathcal{R},p}\right) \tag{26}$$

The similar equality can be checked for the kernels of adjoint operators. Applying Theorem 8.2 from [22], we get

$$\dim \operatorname{coker}\left(P; H_q^{k_1,\mathcal{R},p}\right) = \dim \operatorname{coker}\left(P; H_q^{k_2,\mathcal{R},p}\right) \tag{27}$$

From (26) and (27) we get

$$\operatorname{ind}\left(P; H_q^{k_1,\mathcal{R},p}\right) = \operatorname{ind}\left(P; H_q^{k_2,\mathcal{R},p}\right).$$

□

The results from Theorems 3.2, 3.3, 3.5 and Corollary 3.2 can be summarized in the following way:

Theorem 3.6 *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be the differential form* (4) *with the coefficients that satisfy $\lim\limits_{|x|\to\infty} \max\limits_{|x-y|\le 1} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathcal{R}$. Then the following statements are equivalent:*

1. *Operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ is a Fredholm operator;*

2. *There exist constants $C > 0$ and $N > 0$ such that:*

$$\|u\|_{k+1,\mathcal{R},p,q} \le C\left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_N)}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$

3. *$P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that*

$$\left|\sum_{\alpha\in\mathcal{R}} a_\alpha^0(x) \lambda^{1-\max\limits_j(\alpha:\mu^j)} \xi^\alpha\right| \ge \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Whenever the conditions hold, the kernel, the cokernel dimensions and the index of operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ do not depend on the choice of k.

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