Bloch spectral analysis in the class of non-periodic laminates

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Abstract. In this work, we introduce Bloch waves to study the homogenization process in a class of simple laminates which are obtained as a particular Hashin-Shtrikman microstructure involving translations and dilations in only one direction. This makes this class of microstructures non necessarily periodic in the direction of lamination. We derive explicit formulae for the Bloch wave spectral representation of the homogenized coefficients.

Keywords: Homogenization, Laminates, Hashin-Shtrikman, Spectral analysis, Bloch waves.

1 Introduction

This paper deals with Bloch wave spectral analysis of a class of non-periodic simple laminates. This work is in the same spirit as our earlier work [1], where we perform the Bloch wave spectral analysis in the class of generalized (non-periodic) Hashin-Shtrikman microstructures. Here we are interested in the Bloch spectral representation of the homogenized coefficients, where the inhomogeneities are governed with both non-uniform scales and transformations in one direction and maintains uniformity with respect to scales and translations in other directions, which concerns both Hashin-Shtrikman and laminates. As a special case, this also includes the case of periodic laminates.

Let us start recalling the definition of *G-convergence*. To this end, we consider a sequence of symmetric matrices $A^n \in \mathcal{M}(\alpha,\beta;\Omega)$, for some $0 < \alpha < \beta$, Ω a given open bounded subset

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of $\mathbb{R}^N$, and

$$\mathcal{M}(\alpha,\beta;\Omega) = \Big\{A(x) = [a_{kl}(x)] : a_{kl}(x) = a_{lk}(x)\ \forall k,l \in \{1,\dots,N\},\\ \alpha|\xi|^2 \le A(x)\xi\cdot\xi \le \beta|\xi|^2,\ \text{ for any } \xi \in \mathbb{R}^N,\ \text{a.e. } x \in \Omega\Big\}. \quad (1)$$

Here, we have denoted by $\cdot$ the inner product in $\mathbb{R}^N$.

(A^n) is said to converge in the sense of homogenization to a homogenized limit (G-limit) matrix $A^* \in \mathcal{M}(\alpha,\beta;\Omega)$, i.e. $A^n \xrightarrow{G} A^*$, if for any right hand side $f \in H^{-1}(\Omega)$, the sequence u^n, solution of the problem

$$\begin{aligned} \mathcal{A}^n u^n(x) \overset{\text{(def)}}{=} -\operatorname{div}(A^n(x)\nabla u^n(x)) &= f(x) \quad \text{in } \Omega,\\ u^n(x) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

satisfies

$$\begin{aligned} u^n(x) &\rightharpoonup u(x) && \text{weakly in } H_0^1(\Omega),\\ A^n(x)\nabla u^\varepsilon(x) &\rightharpoonup A^*(x)\nabla u(x) && \text{weakly in } (L^2(\Omega))^N, \end{aligned}$$

where u is the solution of the homogenized equation

$$\begin{aligned} -\operatorname{div}(A^*(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega,\\ u(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The homogenized limit A^* is locally defined and does not depend on the source term f or the boundary condition on $\partial\Omega$.

Laminated microstructure: Let us now introduce the so-called layers or laminated materials. They are defined as those in which the properties and geometry of the medium vary in only one direction, in the sense that the sequence of matrices A^n depends on a single space variable, say $A^n = A^n(x\cdot e_r)$ (where e_r is some standard unit vector in $\mathbb{R}^N$). The corresponding composite is called a laminate, generalizing the one-dimensional settings. In particular, if $A^n \in \mathcal{M}(\alpha,\beta;\Omega)$ satisfies the assumption

$$A^n(x) = a^n(x\cdot e_r)I \ \text{ a.e. } x \in \Omega,$$

the standard G-convergence concept reduces to the usual weak convergence of some combinations of entries of the matrix A^n (here, I denotes the identity matrix). Indeed, this yields another type of explicit formula for the homogenized matrix as in the one-dimensional case: A^n G-converges to the homogenized matrix A^* if and only if the following convergences hold in $L^\infty(\Omega)-$weak* (see [2]):

$$\frac{1}{A^n_{rr}} \rightharpoonup \frac{1}{A^*_{rr}} \ \text{ and } A^n_{jj} \rightharpoonup A^*_{jj} \ \text{ for } 1 \le j \le N \text{ and } j \ne r \ \text{ weakly* in } L^\infty(\Omega),$$

where A^*_{jj} is the arithmetic mean of $a^n(x\cdot e_r)$ for $1 \le j \le N$, $j \ne r$ and A^*_{rr} is the harmonic mean of $a^n(x\cdot e_r)$.

Hashin-Shtrikman microstructures: Next, we introduce the Hashin-Shtrikman microstructures and its G-limit. In his book [3, page no. 281], L. Tartar introduces the notion of a homogeneous medium being equivalent to the micro-structured medium.

Definition 1.1 (Microstructure equivalent to a constant matrix M**)** *Let $\omega \subset \mathbb{R}^N$ be an open connected bounded set with Lipschitz boundary. Let $M = [m_{kl}]_{1\le k,l\le N}$ be a constant symmetric positive definite $N\times N$ matrix.*

We say that a microstructure $A_\omega(y) = [a^{\omega}_{kl}(y)]_{1\le k,l\le N} \in \mathcal{M}(\alpha,\beta;\omega)$ *(see* (1)*) is equivalent to* M *if after extending* A_ω *by* M *in* $\mathbb{R}^N \setminus \omega$*, it follows that for any* $\lambda \in \mathbb{R}^N$ *there exists* $w_\lambda \in H^1_{loc}(\mathbb{R}^N)$ *satisfying*

$$\begin{cases} -\mathrm{div}(A_\omega(y)\nabla w_\lambda(y)) &= 0 \quad \text{in } \mathbb{R}^N, \\ w_\lambda(y) &= \lambda\cdot y \quad \text{in } \mathbb{R}^N \setminus \omega. \end{cases} \tag{2}$$

Remark 1.1 *The function* w_λ *defined in* (2) *can be equivalently obtained solving the following problem in* ω*:*

$$\begin{aligned} -\mathrm{div}(A_\omega \nabla w_\lambda(y)) = 0 \text{ in } \omega, \\ w_\lambda(y) - \lambda\cdot y \in H^1_0(\omega), \end{aligned} \tag{3}$$

with the additional boundary condition

$$A_\omega \nabla w_\lambda \cdot \nu = M\lambda\cdot\nu \text{ on } \partial\omega. \tag{4}$$

We remark that, as viewed in this manner on ω*, the above system is overdetermined since there exist too many boundary conditions. Indeed, the solution of problem* (3) *satisfies condition* (4) *if and only if* A_ω *is equivalent to the matrix* M*.*

Definition 1.2 (Sequence of Vitali's coverings of Ω) *Let* $\omega, \Omega \subset \mathbb{R}^N$ *be two bounded open sets with Lipschitz boundaries. A sequence (indexed by* $n \in \mathbb{N}$*) of Vitali's coverings of* Ω *by reduced and disjoint copies of* ω*, consists of the sequence of coverings* $\{\bigcup_{p\in K_n}(\varepsilon_{p,n}\omega + y^{p,n})\}_{n\in\mathbb{N}}$*, where the sets* K_n *are finite or countable, such that for any* $n \in \mathbb{N}$ *we have*

$$(\varepsilon_{p,n}\omega + y^{p,n}) \cap (\varepsilon_{q,n}\omega + y^{q,n}) = \emptyset \qquad \forall p,q \in K_n, p \neq q, \tag{5}$$

and

$$meas(\Omega \setminus \bigcup_{p\in K_n}(\varepsilon_{p,n}\omega + y^{p,n})) = 0. \tag{6}$$

Moreover, if we define $\kappa_n = \sup_{p\in K_n} \varepsilon_{p,n}$*, we have* $\kappa_n \to 0$ *as* $n \to \infty$*.*

Definition 1.3 *Let us consider a sequence of Vitali's coverings of* Ω *given in Definition 1.2, and a microstructure* A_ω *which we assume to be equivalent to a given matrix* M *in the sense of Definition 1.1. Using* (5)–(6)*, it is clear that for any* $n \in \mathbb{N}$ *and for almost every* $x \in \Omega$*, there exists a unique* $p \in K_n$ *(depending on* x*) such that* $x \in \varepsilon_{p,n}\omega + y^{p,n}$ *and* $\frac{x-y^{p,n}}{\varepsilon_{p,n}} \in \omega$*. Then, we define*

$$A^n_\omega(x) = A_\omega\Big(\frac{x-y^{p,n}}{\varepsilon_{p,n}}\Big), \quad p \in K_n \text{ such that } x \in \varepsilon_{p,n}\omega + y^{p,n}, \text{ for a.e. } x \in \Omega. \tag{7}$$

This construction (7) *represents the so-called* ***Hashin-Shtrikman microstructures*** *in* Ω*.*

An important convergence property for this kind of materials comes from the following analysis: If, for a given $\lambda \in \mathbb{R}^N$, we define $v^n_\lambda \in L^2(\Omega)$ by

$$v^n_\lambda(x) = \varepsilon_{p,n} w_\lambda\Big(\frac{x-y^{p,n}}{\varepsilon_{p,n}}\Big) + \lambda\cdot y^{p,n}, \quad p \in K_n \text{ such that } x \in \varepsilon_{p,n}\omega + y^{p,n}, \text{ for a.e. } x \in \Omega, \tag{8}$$

where w_λ is defined in (2), then, one has the following properties (see [3, Page no. 283]):

$$\begin{cases} v^n_\lambda \in H^1(\Omega), \\ -\mathrm{div}(A^n_\omega(x)\nabla v^n_\lambda(x)) = 0 \quad \text{in } \Omega. \end{cases}$$

Additionally, as $n \to \infty$, one gets the following convergences:

$$v_\lambda^n \rightharpoonup \lambda \cdot x \text{ weakly in } H^1(\Omega),$$
$$A_\omega^n \nabla v_\lambda^n \rightharpoonup M\lambda \text{ weakly in } L^2(\Omega)^N.$$

Since the previous results are valid for any $\lambda \in \mathbb{R}^N$, by the definition of G-convergence, one can easily check that the entire sequence (A_ω^n) satisfies

$$A_\omega^n \xrightarrow{G} M.$$

One relevant thing to be noticed is that the G-limit does not depend on the choice of translations $y^{p,n}$ and scales $\varepsilon_{p,n}$, as long as they are imposed to satisfy the Vitali's covering criteria (5) and (6).

Let us give two examples of microstructures equivalent to a constant matrix.

Example 1.1 (Spherical inclusions in two-phase medium) *Let* $\omega = B(0,1) = \{y : |y| < 1\}$ *and* $R \in (0,1)$*. We consider the microstructure*

$$A_\omega(y) = \begin{cases} \alpha I & \text{if } |y| \le R, \\ \beta I & \text{if } R < |y| \le 1, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ *are known as core and coating conductivities, respectively. Then,* A_ω *is equivalent to* γI*, where* γ *satisfies (see [3]):*

$$\frac{\gamma - \beta}{\gamma + (N-1)\beta} = \theta \frac{\alpha - \beta}{\alpha + (N-1)\beta}, \quad \text{with } \theta = R^N.$$

Example 1.2 (Elliptical inclusions in two-phase medium) *For* $m_1, \dots, m_N \in \mathbb{R}$ *and* $\rho \in (-m, \infty)$*, where* $m = \min\{m_j : j = 1, \dots, N\}$*, the family of confocal ellipsoids* S_ρ *are defined by the equation*

$$\sum_{j=1}^N \frac{y_j^2}{\rho + m_j} = 1.$$

Given $\rho_1, \rho_2 \in (-m, \infty)$ *such that* $\rho_1 < \rho_2$*, let us consider* $\omega = E_{\rho_2+m_1,\dots,\rho_2+m_N} = \left\{y : \sum_{j=1}^N \frac{y_j^2}{\rho_2+m_j} \le 1\right\}$, $\omega_I = E_{\rho_1+m_1,\dots,\rho_1+m_N} = \left\{y : \sum_{j=1}^N \frac{y_j^2}{\rho_1+m_j} \le 1\right\}$ *and the material defined by*

$$A_\omega(y) = a_E(y) I = \begin{cases} \alpha I & \text{if } y \in \omega_I, \\ \beta I & \text{if } y \in \omega \setminus \omega_I, \end{cases}$$

then A_ω *is equivalent to a constant diagonal matrix* $\Gamma = [\gamma_{jj}]_{1 \le j \le N}$ *(see [3, Corollary 26.4]) given by*

$$\frac{1}{\beta - \gamma_{jj}} = \frac{V(\rho_2)}{(\beta - \alpha)V(\rho_1)} - \frac{V(\rho_2)}{2\beta} \int_{\rho_1}^{\rho_2} \frac{d\rho}{(\rho + m_j)V(\rho)}, \quad \text{where } V(\rho) = \prod_k \sqrt{\rho + m_k}.$$

Following its construction, the coefficients in Hashin-Shtrikman microstructures are invariant in a certain way of both translations and dilations of the medium. Classical periodic microstructures incorporate uniform translations and uniform dilations with respect to only one scale ε, whereas Hashin-Shtrikman construction incorporates non-uniform translations and dilations with a family of scales $\{\varepsilon_p\}_p$. It is a non-periodic and non-commutative class of microstructures, where we would like to develop the Bloch wave spectral analysis. The importance of HS microstructures and their role in homogenization theory is well-known. They are among extreme points of the so-called G-closure set/phase diagram of mixtures of two-phase materials in a prescribed volume proportion. They are also solutions of optimal design problems (see [2, 3]).

Let us now define the subclass of non-periodic laminates, contained as a particular case of the generalized Hashin-Shtrikman microstructures, which we are interested in studying. We will refer to them as HS laminates.

A subclass of HS laminates in two-phase medium: Let us consider $S = [-1,1]^N$, and $\theta \in (0,1)$. For some fixed $r \in \{1,\dots,N\}$, we define $A_S \in \mathcal{M}(\alpha,\beta;S)$ as follows

$$A_S(y) = a_S(y\cdot e_r)I = \begin{cases} \alpha I & \text{whenever } y\cdot e_r \in [-\theta,\theta] \subset [-1,1], \\ \beta I & \text{elsewhere.} \end{cases} \tag{9}$$

The parameter θ represents the volume fraction of material α.

We now seek a positive constant m such that $a_S(y\cdot e_r)$ is equivalent to m in the e_r-direction. To do this, we extend $a_S(y\cdot e_r)$ by m for $y\cdot e_r \in \mathbb{R}\setminus[-1,1]$ and we study the existence of function $w_{e_r} \in H^1_{loc}(\mathbb{R})$ satisfying:

$$-\frac{d}{dy_r}\Big(a_S(y_r)\frac{dw_{e_r}}{dy_r}(y_r)\Big) = 0 \quad \text{in } \mathbb{R}, \quad w_{e_r}(y_r) = y_r \quad \text{in } \mathbb{R}\setminus[-1,1]. \tag{10}$$

From Example 1.1, restricting it to the one-dimensional case ($N = 1$), we establish the existence of $w_{e_r} \in H^1_{loc}(\mathbb{R})$ satisfying (10), insofar m is given by

$$m = \frac{\alpha\beta}{\theta\beta + (1-\theta)\alpha} \quad \text{(the harmonic mean of } \alpha \text{ and } \beta \text{ with proportion } \theta).$$

In the other directions e_j ($j \in \{1,\dots,r-1,r+1,\dots,N\}$), we define the microstructure by using periodic arrays in the following way:

i) We define $S_\perp$ as the extension of the domain S by periodicity in all directions e_j, $j \neq r$, that is $S_\perp = \bigcup_{\substack{z\in\mathbb{Z}^N \\ z_r=0}} (S + 2z)$.

ii) We use a special sequence of Vitali's coverings $(\varepsilon_{p,n}, y^{p,n})$ of Ω by reduced and disjoint copies of $S_\perp$ in the e_r-direction such that, for any $n \in \mathbb{N}$, we have

$$(\varepsilon_{p,n}S_\perp + y^{p,n}) \cap (\varepsilon_{q,n}S_\perp + y^{q,n}) = \emptyset \qquad \forall p,q \in K_n, p \neq q, \tag{11}$$

and

$$meas\,(\Omega \setminus \bigcup_{p\in K_n}(\varepsilon_{p,n}S_\perp + y^{p,n})) = 0, \text{ with } \kappa_n = \sup_{p\in K_n} \varepsilon_{p,n} \to 0, \tag{12}$$

for a finite or countable K_n. These define the microstructures in A^n_S as follows:

$$A^n_S(x) = a_S\Big(\frac{x - y^{p,n}}{\varepsilon_{p,n}}\cdot e_r\Big)I \text{ in } \varepsilon_{p,n}S_\perp + y^{p,n} \text{ a.e. in } \Omega, \quad p \in K_n, \tag{13}$$

which makes sense since, for each n, the sets $\varepsilon_{p,n}S_\perp + y^{p,n}$, $p \in K$, are disjoint. The above construction (13) represents one subclass of *non-periodic laminate* microstructures in two-phase medium.

Consequently, one has the following G-convergence of the entire sequence

$$A_S^n \xrightarrow{G} A_S^* = \ diag(\overline{a_S}, \ldots, \underline{a_S}, \ldots, \overline{a_S}), \quad \underline{a_S} \text{ comes at only } r\text{-th diagonal entry,} \tag{14}$$

where we have denoted by $\overline{a_S}$ and $\underline{a_S}$ the arithmetic mean and harmonic mean of a_S, respectively.

Due to the periodicity in the e_j-directions ($j \neq r$), the homogenized conductivity $(A_S^*)_{jj}$ ($j \neq r$) is defined by its entries:

$$(A_S^*)_{jj} = \frac{1}{|S|} \int_S a_S(y \cdot e_r)(\nabla \chi_j + e_j) \cdot (\nabla \chi_j + e_j) dy, \quad j \in \{1, \ldots, r-1, r+1, \ldots, N\}, \tag{15}$$

where, for each $j \neq r, \chi_j \in H^1(S)$ solves the following cell problem

$$-\text{div}\,(a_S(y \cdot e_r)(\nabla \chi_j(y) + e_j)) = 0 \ \text{ in } S, \ y \mapsto \chi_j(y) \ \text{ is } 1\text{−periodic in each } e_j\text{-direction} \ \ (j \neq r). \tag{16}$$

As we see, $\chi_j(y) = 0$ ($j \neq r$) uniquely solves the above equation to give

$$(A_S^*)_{jj} = \theta\alpha + (1 - \theta)\beta = \overline{a} \text{ (the arithmetic mean of } \alpha \text{ and } \beta \text{ with proportion } \theta \text{).}$$

Thus, the limit in (14) is well understood now.

Remark 1.2 *As we see, the microstructures governed by* (13) *are periodic in* ($N - 1$) *directions and in one direction it includes one-dimensional Hashin-Shtrikman construction.*

We would like now to perform a Bloch spectral analysis for this class of microstructures. Let us briefly explain what is behind Bloch spectral analysis. Following the classical Fourier approach for homogeneous media, one tries to diagonalize the operator and introduces Bloch waves (BW) as its eigenvectors in this approach. One invokes the Floquet approach to solve a parameter-dependent eigenvalue problem; namely, periodic media are multiplicative perturbations of homogeneous media. This gives rise to a generalized periodicity condition for Bloch waves. The homogenized matrix and the oscillatory test functions are obtained as infinitesimal versions of BW and its eigenvalue (energy), which lie at the lower end of the spectrum. It is found that BW (as well as its eigenvalues) carry a discrete (energy) index and a continuous (quasi-momentum) index. The significance lies in the fact that the corresponding eigenvalues represent different energy levels and their discreteness brings some simplification. Another nice feature is the regularity of the lower BW and their energy with respect to the momentum variables. It is no surprise that these two properties play an important role in the homogenization process.

As far as we know, the problem of obtaining a BW-type basis is open to arbitrary microstructures, mainly because there might not exist any associated invariance. It is now appropriate to recall that we only need the lower part of the spectrum for homogenization, and even that seems to be an open problem. However, the BW method has already been successfully applied to periodic microstructures. This approach is also known as Bloch wave homogenization (or spectral approach), some references are [4–8]. We must note that the BW method defines a higher order approximation than homogenization. More precisely, exploiting the regularity of the BW fundamental eigenvalue with respect to the momentum variables, it is possible to define a dispersion approximation for periodic and HS media [9–11]. In general, the development of the BW approach is open for non-periodic microstructures. It is also used for the non-periodic generalized Hashin-Shtrikman microstructures in paper [1] and for the case of quasi-periodic media in [12]. The present work is a contribution in this direction. More precisely, we develop the BW method for the HS laminates defined above.

2 Bloch spectral analysis

We take $A_S(y) = a_S(y \cdot e_r)I \in \mathcal{M}(\alpha,\beta;S)$ defined in (9) to consider the following spectral problem parameterized by $\eta \in \mathbb{R}^N$: Find $\mu := \mu(\eta) \in \mathbb{C}$ and $\varphi_S := \varphi_S(y;\eta)$ (not identically zero) such that

$$\begin{cases} \mathcal{A}_S(\eta)\varphi_S(y;\eta) \overset{\text{(def)}}{=} -\Big(\dfrac{\partial}{\partial y_k} + i\eta_k\Big)\Big(a_S(y\cdot e_r)\Big(\dfrac{\partial}{\partial y_k} + i\eta_k\Big)\Big)\varphi_S(y;\eta) = \mu(\eta)\varphi_S(y;\eta) \quad \text{in } S, \\ \qquad \varphi_S(y;\eta) \text{ is constant on } \{y\cdot e_r = \pm 1\}, \\ \qquad \varphi_S(y;\eta) \text{ is 2-periodic in each } e_j\text{-direction } (j \neq r), \\ \displaystyle\int_{\{y\cdot e_r = \pm 1\}} a_S(y\cdot e_r)\Big(\frac{\partial}{\partial y_k} + i\eta_k\Big)\varphi_S(y;\eta)\nu_k \, d\sigma = 0, \end{cases} \tag{17}$$

where $\nu = \pm e_r$ is the outer normal unit vector on the boundary $\{y \cdot e_r = \pm 1\}$ and $d\sigma$ is the surface measure on $\{y \cdot e_r = \pm 1\}$.

Weak formulation: We first introduce the function spaces

$$L^2_{c,\#}(S) = \Big\{\varphi \in L^2_{loc}(\mathbb{R}^N) \; : \varphi \text{ is constant when } y\cdot e_r \in \mathbb{R}\setminus(-1,1) \text{ and } \varphi \text{ is 2-periodic in the } e_j\text{-direction } (j\neq r)\Big\},$$

$$H^1_{c,\#}(S) = \Big\{\varphi \in H^1_{loc}(\mathbb{R}^N) \; : \varphi \text{ is constant when } y\cdot e_r \in \mathbb{R}\setminus(-1,1) \text{ and } \varphi \text{ is 2-periodic in the } e_j\text{-direction } (j\neq r)\Big\}.$$

Here, c is a floating constant depending on the element under consideration. $L^2_{c,\#}(S)$ and $H^1_{c,\#}(Y)$ are subspace of $L^2(S)$ and $H^1(S)$ respectively, the second inclusion being proper. They inherit the subspace norm-topology of the parent space.

The motivation for the state space $H^1_{c,\#}$ starting from $a_S(y \cdot e_r)$ is equivalent to m in the e_r-direction, but independent of other y_j variables $(j \neq r)$. So, in particular, in other directions it is periodic with any period, for convenience we take 2−periodicity (the size of the edges of the reference cell S).

Similarly, one can define $L^2_{c,\#}(\eta;S)$ or $H^1_{c,\#}(\eta;S)$ spaces as follows:

$$L^2_{c,\#}(\eta;S) = \Big\{\varphi \in L^2_{loc}(\mathbb{R}^N) \; : e^{-iy\cdot\eta}\varphi \text{ is constant when } y\cdot e_r \in \mathbb{R}\setminus(-1,1) \text{ and } e^{-iy\cdot\eta}\varphi \text{ is } 2-\text{periodic in the } e_j\text{-direction } (j\neq r)\Big\},$$

$$H^1_{c,\#}(\eta;S) = \Big\{\varphi \in H^1_{loc}(\mathbb{R}^N) \; : e^{-iy\cdot\eta}\varphi \text{ is constant when } y\cdot e_r \in \mathbb{R}\setminus(-1,1) \text{ and } e^{-iy\cdot\eta}\varphi \text{ is } 2-\text{periodic in the } e_j\text{-direction } (j\neq r)\Big\}.$$

As a next step, we give the weak formulation of the problem (17) in these function spaces. We are interested in proving the existence of eigenvalues and their corresponding eigenvectors $(\mu(\eta), \varphi_S(y;\eta))$ with $\mu(\eta) \in \mathbb{C}$ and $\varphi_S(\cdot;\eta) \in H^1_{c,\#}(S)$ of

$$a_S(\eta)(\varphi_S(y;\eta),\psi) = \mu(\eta)(\varphi_S(y;\eta),\psi) \quad \forall \psi \in H^1_{c,\#}(S), \tag{18}$$

where the bilinear forms $a_S(\eta)(\cdot,\cdot)$ and $(\cdot,\cdot)$ are defined by

$$\begin{aligned} a_S(\eta)(u,w) &= \int_S a_S(y\cdot e_r)(y)\left(\frac{\partial u}{\partial y_k} + i\eta_k u\right)\overline{\left(\frac{\partial w}{\partial y_k} + i\eta_k w\right)}dy, \\ (u,w) &= \int_S u\overline{w}dy. \end{aligned}$$

Existence Result: By following the same analysis presented in [1], we state the corresponding existence result for the problem (18).

Proposition 2.1 *Fix $\eta \in \mathbb{R}^N$. Then, there exist a sequence of eigenvalues $\{\mu_m(\eta) \ : \ m \in \mathbb{N}\}$ and there corresponding eigenvectors $\{\varphi_{S,m}(y;\eta) \in H^1_{c,\#}(S) \ : \ m \in \mathbb{N}\}$ such that*

$$a_S(\eta)(\varphi_{S,m}(y;\eta), \psi) = \mu_m(\eta)(\varphi_{S,m}(y;\eta), \psi) \quad \forall \psi \in H^1_{c,\#}(S) \ \text{and} \ \forall m \in \mathbb{N}.$$

Regularity of the ground state: In the next proposition, we establish a regularity result for the ground state on the basis of Kato-Rellich spectral analysis. The proof can be found in [1], where the following notation is used.

Notation 2.1 *For integers $k, l \in \{1, \ldots, N\}$, we use the symbols D_k, $D^2_{kl}, \ldots$ to denote the derivatives $\frac{\partial}{\partial \eta_k}$, $\frac{\partial^2}{\partial \eta_k \partial \eta_l}, \ldots$ respectively. Additionally, for a given multi-index $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots, \ell_N) \in \mathbb{Z}^N_+$ ($\boldsymbol{\ell} \neq \mathbf{0}$), we use the symbol $D^{|\boldsymbol{\ell}|}_{\boldsymbol{\ell}}$ to denote the derivative $\frac{\partial^{|\boldsymbol{\ell}|}}{\partial \eta_1^{\ell_1} \cdots \partial \eta_N^{\ell_N}}$, where $|\boldsymbol{\ell}| = \ell_1 + \cdots + \ell_N$.*

Proposition 2.2

1. *Zero is the first eigenvalue of* (18) *at $\eta = 0$ and it is an isolated point of the spectrum with its algebraic multiplicity equals to one.*
2. *There exists an open neighborhood S' around zero such that the first eigenvalue $\mu_1(\eta)$ is an analytic function on S' and there is a choice of the first eigenvector $\varphi_{S,1}(y;\eta)$ satisfying*

$$\eta \mapsto \varphi_{S,1}(\cdot;\eta) \in H^1_{c,\#}(S) \ \text{is analytic on} \ S' \ \text{and} \ \varphi_{S,1}(y;0) = |S|^{-1/2},$$

with the boundary normalization condition $D^{|\boldsymbol{\ell}|}_{\boldsymbol{\ell}} \varphi_{S,1}(y;0) = 0$ on the boundary $\{y \cdot e_r = \pm 1\}$, for $\boldsymbol{\ell} \in \mathbb{Z}^N_+ \setminus \{0\}$.

Derivatives of $\mu_1(\eta)$ and $\varphi_{S,1}(\eta)$ at $\eta = 0$: The procedure consists of differentiating the eigenvalue equation (17) for $\mu(\eta) = \mu_1(\eta)$ and $\varphi_S(\cdot;\eta) = \varphi_{S,1}(\cdot;\eta)$.

Step 1. Zeroth order derivatives: We simply recall that $\varphi_{S,1}(y;0) = |S|^{-1/2}$ by our choice and $\mu_1(0) = 0$.

Step 2. First order derivatives of $\mu_1(\eta)$ at $\eta = 0$: By differentiating once the first equation in (17) with respect to η_k, we obtain

$$D_k(\mathcal{A}_S(\eta) - \mu_1(\eta))\varphi_{S,1}(\cdot;\eta) + (\mathcal{A}_S(\eta) - \mu_1(\eta))(D_k\varphi_{S,1}(\cdot;\eta)) = 0, \tag{19}$$

and, for $\eta = 0$, since $D_k\mathcal{A}_S(0)\varphi_{S,1}(\cdot;0) = iC^S_k\varphi_{S,1}(\cdot;0)$, where $C^S_k\varphi = -a_S(y \cdot e_r)\frac{\partial \varphi}{\partial y_k} - \frac{\partial}{\partial y_k}(a_S(y \cdot e_r)\varphi)$, we deduce that

$$(iC^S_k - (D_k\mu_1)(0))\varphi_{S,1}(\cdot;0) + \mathcal{A}_S(D_k\varphi_{S,1})(\cdot;0) = 0. \tag{20}$$

Taking scalar product with $\varphi_{S,1}(\cdot;0)$ in $L^2(S)$, integrating by parts and using the fact that $\varphi_{S,1}(\cdot;0) = |S|^{-1/2}$, it follows therefore that

$$D_k\mu_1(0) = 0 \quad \forall\, k = 1, \ldots, N. \tag{21}$$

Step 3. First order derivatives of $\varphi_{S,1}(\cdot;\eta)$ at $\eta = 0$: Combining (21)–(20), and differentiating the boundary condition in (17) with respect to η_k, one has

$$-\frac{\partial}{\partial y_l}\Big(a_S(y\cdot e_r)\frac{\partial}{\partial y_l}D_k\varphi_{S,1}(\cdot;0)\Big) = -iC_k^S\varphi_{S,1}(\cdot;0) = i\varphi_{S,1}(\cdot;0)\frac{\partial}{\partial y_k}(a_S(y\cdot e_r)) \quad \text{in } S, \tag{22}$$

$$D_k\varphi_{S,1}(\cdot;0) = 0 \quad \text{on } \{y\cdot e_r = \pm 1\}, \tag{23}$$

$$D_k\varphi_{S,1}(\cdot;0) \text{ is 2-periodic in the } e_j\text{-direction } (j\neq r), \tag{24}$$

$$\text{and} \quad \int_{\{y\cdot e_r=\pm 1\}} a_S(y\cdot e_r)\Big(\nabla_y D_k\varphi_{S,1}(y;0) + i\varphi_{S,1}(y;0)e_k\Big)\cdot\nu\, d\sigma = 0. \tag{25}$$

For $k = r$, we seek $D_r\varphi_{S,1}(y;0) = \dfrac{1}{i}\dfrac{d}{dy_r}\varphi_{S,1}(y_r;0)$ solving the equation (22) uniquely with the Dirichlet boundary condition (23). In that case, (22) becomes an ordinary differential equation and gets identified with (10) to give

$$\frac{1}{i}\frac{d}{dy_r}\varphi_{S,1}(y_r;0) = i|S|^{-1/2}(w_{e_r}(y_r) - y_r)$$

satisfying (25) also.

For $k \neq r$, we notice that $D_k\varphi_{S,1}(y;0) = 0$ is the unique solution of the above system of equations.

Step 4. Second derivatives of $\mu_1(\eta)$ at $\eta = 0$: We differentiate (19) with respect to η_k and then, by taking scalar product with $\varphi_{S,1}(\cdot;\eta)$ in $L^2(S)$, one obtains

$$\langle D^2_{kk}(\mathcal{A}_S(\eta)-\mu_1(\eta))\varphi_{S,1}(\cdot;\eta),\varphi_{S,1}(\cdot;\eta)\rangle + 2\langle[D_k(\mathcal{A}_S(\eta)-\mu_1(\eta))]D_k\varphi_{S,1}(\cdot;\eta),\varphi_{S,1}(\cdot;\eta)\rangle = 0.$$

Evaluating at $\eta = 0$ and using the computation from the previous steps, it simply becomes

$$\frac{1}{2}D^2_{ii}\mu_1(0) = \frac{1}{|S|}\int_S a_S(y\cdot e_r)dx - \frac{1}{|S|}\int_S C_r^S(w_{e_r}(y_r)-y_r)dy = m = \underline{a_S},$$

$$\text{and} \quad \frac{1}{2}D^2_{jj}\mu_1(0) = \frac{1}{|S|}\int_S a_S(y\cdot e_r)dx = \overline{a_S} \quad \forall j\neq r,$$

which are indeed the homogenized coefficients governed with the simple laminates in two-phase medium stated in (14).

As a next step, one can define the first Bloch transform likewise in [1] and successively perform the limit analysis to obtain the homogenization result as derived in [1]. We briefly mention those results here.

3 Bloch waves at $\varepsilon_{p,n}$–scales, $y^{p,n}$–translations and Bloch transform

Motivated by the HS construction (13), we introduce the operator:

$$\mathcal{A}_S^n \overset{\text{(def)}}{=} -\frac{\partial}{\partial x_k}\Big(a_S\Big(\frac{x-y^{p,n}}{\varepsilon_{p,n}}\cdot e_r\Big)\frac{\partial}{\partial x_k}\Big), \quad x\in\varepsilon_{p,n}S_\perp + y^{p,n}, \tag{26}$$

and its shifted version

$$\mathcal{A}_S^n(\xi) \overset{\text{(def)}}{=} -\Big(\frac{\partial}{\partial x_k}+i\xi_k\Big)\Big(a_S\Big(\frac{x-y^{p,n}}{\varepsilon_{p,n}}\cdot e_r\Big)\Big(\frac{\partial}{\partial x_k}+i\xi_k\Big)\Big), \quad x\in\varepsilon_{p,n}S_\perp + y^{p,n}. \tag{27}$$

By homothecy, for a fixed n and for each p, we define the first Bloch eigenvalue $\mu_1^{n,p}(\xi)$ and the corresponding Bloch mode $\varphi_{S,1}^{n,p}(\cdot;\xi)$ for the operator $(\mathcal{A}_S^n)(\xi)$ for $\xi \in \kappa_n^{-1}S'$ as follows:

$$\mu_1^{n,p}(\xi) \overset{(\mathrm{def})}{=} \varepsilon_{p,n}^{-2}\mu_1(\varepsilon_{p,n}\xi), \quad \varphi_{S,1}^{n,p}(x;\xi) \overset{(\mathrm{def})}{=} \varphi_{S,1}\Big(\frac{x - y^{p,n}}{\varepsilon_{p,n}};\varepsilon_{p,n}\xi\Big) \quad \text{for } x \in \varepsilon_{p,n}S_\perp + y^{p,n}, \tag{28}$$

where $\mu_1(\eta)$ and $\varphi_{S,1}(y;\eta)$ are the eigenelements defined in Section 2.

By grouping these Bloch modes $\varphi_{S,1}^{n,p}(\cdot;\xi)$, we can define a global function $\varphi_{S,1}^{n}(\cdot;\xi)$ in the entire domain Ω as follows: For a.e. $x \in \Omega$ there exists a unique $p = p(x) \in K_n$ such that $x \in \varepsilon_{p,n}S_\perp + y^{p,n}$ and $\frac{x-y^{p,n}}{\varepsilon_{p,n}} \in S_\perp$. Then, for a.e. $x \in \Omega$, we define

$$\varphi_{S,1}^{n}(x;\xi) = \varphi_{S,1}^{n,p}(x;\xi) = \varphi_{S,1}\Big(\frac{x - y^{p,n}}{\varepsilon_{p,n}};\varepsilon_{p,n}\xi\Big) \quad \text{for } x \in \varepsilon_{p,n}S_\perp + y^{p,n}, \quad p \in K_n. \tag{29}$$

We now define the Bloch transform in $L^2(\mathbb{R}^N)$:

Definition 3.1 *For $g \in L^2(\Omega)$ and for each $n \in \mathbb{N}$, we define the first Bloch transform of g, governed by the Hashin-Shtrikman microstructures by:*

$$B_1^n g(\xi) := \int_\Omega g(x)e^{-ix\cdot\xi}\overline{\varphi_{S,1}^n}(x;\xi)dx \qquad \forall \xi \in \kappa_n^{-1}S'. \tag{30}$$

The interested reader is referred to [1] for the well-definedness of (30) and the proofs of Proposition 3.1, Proposition 3.2, and Theorem 3.1 below.

Remark 3.1 *For each fixed n, the Bloch transform $B_1^n = B_1^{(\varepsilon_{p,n},\, y^{p,n})_{p\in K_n}}$ depends on the choice of the Vitali's covering. However, in the rest of the paper, we will be using the short notation B_1^n instead of $B_1^{(\varepsilon_{p,n},\, y^{p,n})_{p\in K_n}}$. By that we basically mean the integral* (30) *can be rewritten as follows*

$$B_1^n g(\xi) := \sum_{p\in K_n}\int_{\Omega\cap(\varepsilon_{p,n}S_\perp + y^{p,n})} g(x)e^{-ix\cdot\xi}\overline{\varphi_{S,1}}\Big(\frac{x - y^{p,n}}{\varepsilon_{p,n}};\varepsilon_{p,n}\xi\Big)dx. \tag{31}$$

We state few qualitative properties of our Bloch transform, which are useful in the derivation of the homogenization result.

Proposition 3.1 *Let $g \in H^1(\Omega)$ such that $\mathcal{A}_S^n g \in L^2(\Omega)$, we have: $\forall \xi \in \kappa_n^{-1}S'$,*

$$B_1^n(\mathcal{A}_S^n g)(\xi) = \sum_{p\in K_n}\int_{\Omega\cap(\varepsilon_{p,n}S_\perp + y^{p,n})} \mu_1^{n,p}(\xi)g(x)e^{-ix\cdot\xi}\overline{\varphi_{S,1}^{n,p}}(x;\xi)dx + O(\kappa_n)\|g\|_{H^1(\Omega)}. \tag{32}$$

Proposition 3.2 (First Bloch transform B_1^n tends to Fourier transform) *Let $\rho_n = \kappa_n^{-\frac{2-\delta}{N+2}}$, with $\delta \in (0,1)$ and we denote $\chi_{\rho_n S'}(\cdot)$ the characteristic function of the set $\rho_n S'$.*

1. *If $g^n \rightharpoonup g$ weakly in $L^2(\Omega)$, extending by zero g^n and g outside Ω, it follows that $\chi_{\rho_n S'}(\cdot)B_1^n g^n(\cdot) \rightharpoonup \widehat{g}(\cdot)$ weakly in $L^2(\mathbb{R}^N_\xi)$.*

2. *If $g^n \to g$ strongly in $L^2(\Omega)$, then $\chi_{\rho_n S'}(\cdot)B_1^n g^n(\cdot) \to \widehat{g}(\cdot)$ strongly in $L^2(\mathbb{R}^N_\xi)$. Of course, here g^ε and g are extended by zero outside Ω.*

Remark 3.2 *The factor $\chi_{\rho_n S'}(\xi)$ in the above result was used to extend the relevant functions by zero outside their domain of definition and to take into account a proper decay at infinity of $B_1^n g(\xi)$; this is controlled by the weight ρ_n.*

Homogenization result

We state here our homogenization result. This can be proved by using Bloch waves as introduced at the beginning of this section (for more details, see [1]).

Theorem 3.1 *Let us consider* Ω *be an open bounded subset of* $\mathbb{R}^N$. *We consider the operator* $\mathcal{A}_S^n$ *defined in* (26). *Let* $f \in L^2(\Omega)$ *and* $u^n \in H_0^1(\Omega)$ *be the unique solution of the boundary value problem*

$$\mathcal{A}_S^n u^n = f \quad in\ \Omega.$$

Then there exists $u \in H_0^1(\Omega)$ *such that the sequence* u^n *weakly converges to* u *in* $H_0^1(\Omega)$ *with the following convergence of flux*

$$\sigma_S^n = A_S^n \nabla u^n \rightharpoonup A_S^* \nabla u = \sigma_S \quad weakly\ in\ L^2(\Omega)^N.$$

In particular, the limit u *satisfies the homogenized equation:*

$$-\frac{\partial}{\partial x_k}(A_S^* \frac{\partial}{\partial x_k} u) = f \quad in\ \Omega,$$

where A_S^* *is defined in* (14).

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