The $\Psi-$asymptotic equivalence of the Lyapunov matrix differential equations with modified argument

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Abstract. Using the notion of strict $h-$contraction, existence results for $\Psi-$asymptotic equivalence of two pairs of (Lyapunov) matrix differential equations with modified argument are given.

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1 Introduction

The purpose of this paper is to provide sufficient conditions for $\Psi-$asymptotic equivalence of the $\Psi-$bounded solutions of two Lyapunov matrix differential equations with modified argument

$$Z' = A(t)Z + ZB(t) \tag{1}$$

$$Z' = A(t)Z + ZB(t) + F(t, Z(g(t))),\ t \geq t_0. \tag{2}$$

These conditions can be expressed in the terms of fundamental matrices of the matrix differential equations

$$X' = A(t)X \tag{3}$$

$$Y' = YB(t) \tag{4}$$

and on the function F.

Here, Ψ is a matrix function who allows obtaining a mixed asymptotic behavior for the components of solutions of the matrix differential equations.

History of the problem. In many papers in the field of differential equations have been studied the asymptotic equivalence of the ($\Psi-$)bounded solutions of two (Lyapunov matrix) differential equations. See, for example, [2], [3], [5], [8], [9], [11], [14] and in the references there.

In paper [11], the author consider the vector differential equation

$$x' = A(t)x \tag{5}$$

and the perturbed equation with modified argument

$$y' = A(t)y + f(t, y(g(t))). \tag{6}$$

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The author demonstrated in Theorem 2.1, [11] that under some conditions, for every bounded solution x(t) of equation (5), there exists a unique bounded solution y(t) of equation (6) such that

$$\lim_{t\to\infty} \mid y(t) - x(t) \mid = 0. \tag{7}$$

The present paper extend these results in three directions: from systems of differential equations to Lyapunov matrix differential equations with modified argument, the introduction of the matrix function Ψ that permits to obtaining a mixed asymptotic behavior for the components of solutions and in connection with the belonging to space L^r of solutions.

The main tools used in this paper are a fixed point theorem via strict h-contraction and the technique of variation of constants formula combined with Kronecker product of matrices and Hölder's inequality, which has been successfully applied in various fields of matrix theory. See, for example, the cited papers and the references cited therein.

2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let $\mathbb{R}^d$ be the Euclidean $d-$ dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}^d$, let $\| x \| = \max\{\mid x_1\mid, \mid x_2\mid, ..., \mid x_d\mid\}$ be the norm of x (here, ^{T} denotes transpose).

Let $\mathbb{M}_{d\times d}$ be the linear space of all real $d \times d$ matrices.

For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $\mid A \mid$ by formula $\mid A \mid = \sup\limits_{\|x\|\leq 1} \| Ax \|$. It is well-known that $\mid A \mid = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} \mid a_{ij}\mid\}$.

In this paper, we assume that A and B are continuous $d \times d$ matrices on $\mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $F : \mathbb{R}_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ are continuous functions.

By a solution of equation (2) we mean a continuous differentiable $d \times d$ matrix function $Z(t)$ satisfying the equation (2) for all $t \in \mathbb{R}_+ = [0, \infty)$

Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and

$$\Psi = \text{diag}\ [\Psi_1, \Psi_2, \cdots \Psi_d].$$

A matrix P is said to be a projection if $P^2 = P$.

Definition 2.1 *([7]). A vector function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ is said to be $\Psi-$bounded on $\mathbb{R}_+$ if $\Psi(t)\varphi(t)$ is bounded on $\mathbb{R}_+$ (i.e. there exists $m > 0$ such that $\| \Psi(t)\varphi(t) \| \leq m$, for all $t \in \mathbb{R}_+$).*
Otherwise, is said that the function φ is $\Psi-$unbounded on $\mathbb{R}_+$.

Definition 2.2 *([7]). A matrix function $M : \mathbb{R}_+ \longrightarrow M_{d\times d}$ is said to be $\Psi-$ bounded on $\mathbb{R}_+$ if the matrix function $\Psi(t)M(t)$ is bounded on $\mathbb{R}_+$ (i.e. there exists $m > 0$ such that $\mid \Psi(t)M(t) \mid \leq m$, for all $t \in \mathbb{R}_+$).*
Otherwise, is said that the matrix function M is $\Psi-$ unbounded on $\mathbb{R}_+$.

Remark 2.1 *1. The Definitions extend the definitions of boundedness of vector and matrix functions.*
2. For $\Psi = I_d$, one obtain the notion of classical boundedness (see [4]).
3. It is easy to see that if Ψ and Ψ^{-1} are bounded on $\mathbb{R}_+$, then the $\Psi-$boundedness is equivalent with the classical boundedness.

Definition 2.3 *([1]) Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.
The important rules of calculation of the Kronecker product are given in [1], [10] and Lemma 2.1, [7].

Definition 2.4 *([10]) The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called the vectorization operator.

For important properties and rules of calculation of the $\mathcal{V}ec$ operator, see Lemmas 2.2, 2.3, 2.5, [7].

For "*corresponding Kronecker product system associated with (2)",* see Lemma 2.4, [7].

The Lemmas 2.6 and 2.7, [7], play an important role in the proofs of main results of present paper.

Now, we remember the notions of *strict h-contractions* and *strict comparison functions* which are useful in the proof of our main results.

Definition 2.5 *([12]) The function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function if satisfies the following conditions:*
i). h is continuous,
ii). h is monotone increasing,
iii). $\lim_{n\to\infty} h^n(t) = 0$, for all $t > 0$,
iv). $\lim_{t\to\infty} (t - h(t)) = \infty$.

Let (X, d) be a metric space and $f : X \to X$ an operator.

Definition 2.6 *([12]) The operator f is called a strict h-contraction if satisfies the following conditions:*
i). h is a strict comparison function,
ii). $d(f(x), f(y)) \le h(d(x, y))$, for all $x, y \in X$.

Theorem 2.1 *([12]) Let (X, d) be a complete metric space and $f : X \to X$ a strict h-contraction. Then, f is Picard operator.*

At the end of this section, we recall Lemma 2.1 ([8]) which is useful in the proofs of our main results. This Lemma is a generalization of Lemma 10, [6], Lemma 1, [4], (p. 68) and a Lemma from [13].

Lemma 2.1 *Let U(t) be an invertible $d \times d$ matrix which is a continuous function of t on R_+ and let P a projection, $P \in \mathbb{M}_{d\times d}$.*
Suppose that there exist a continuous function $\varphi : \mathbb{R}_+ \to (0, \infty)$ and the constants $M > 0$ and $p > 1$ such that

$$\int_{t_0}^{t} \left(\varphi(s) \mid \Psi(t)U(t)PU^{-1}(s)\Psi^{-1}(s) \mid\right)^p ds \le M, \text{ for } t \ \ge t_0 \ge 0$$

and $\int_{t_0}^{\infty} \varphi^p(s)ds = \infty$.
Then, there exists a constant $N > 0$ *such that*

$$\mid \Psi(t)U(t)P \mid\leq Ne^{-(pM)^{-1}\int_{t_0}^{t}\varphi^p(s)ds},\ for\ t\ \geq t_0.$$

Consequently, $\lim_{t\to\infty} \mid \Psi(t)U(t)P \mid= 0$.

3 Main results

The purpose of this section is to give sufficient conditions for $\Psi-$ asymptotic equivalence of the $\Psi-$ bounded solutions of two pairs of matrix differential equations, namely (1) - (2) and (3) - (8).

The first result is motivated by the Theorem 2.1, [11].

For $t_0 \geq 0$, we consider the equation (3) and perturbed equation

$$Z' = A(t)Z + F(t, Z(g(t))),\ t \geq t_0, \tag{8}$$

Theorem 3.1 *Suppose that:*
1). There are supplementary projections $P_1, P_2 \in \mathbb{M}_{d\times d}$, *a continuous function* $\varphi : \mathbb{R}_+ \to (0,\infty)$ *that satisfies the condition* $\int_0^{\infty}(\varphi(s))^q ds = +\infty$, $q > 1$, *and a constant* $K > 0$ *such that the fundamental matrix X(t) for the linear matrix differential equation (3) satisfies the condition*

$$\int_{t_0}^{t}[\varphi(s)\left|\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right|]^q ds+$$

$$+\int_{t}^{\infty}[\varphi(s)\left|\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right|]^q ds \leq K,$$

for all $t \geq t_0 \geq 0$, *where* t_0 *is sufficiently large;*
2). There exist a strict comparison function $h : \mathbb{R}_+ \to \mathbb{R}_+$ *and a function* $\lambda(t) \in L^p([t_0,\infty))$, $p > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *such that*

$$\varphi^{-1}(t) \mid \Psi(t)\left(F(t, Z_1) - F(t, Z_2)\right) \mid\leq \lambda(t)h\left(\mid \Psi(t)\left(Z_1 - Z_2\right) \mid\right)$$

for all $t \geq t_0$, $Z_1, Z_2 \in \mathbb{M}_{d\times d}$;
3). $\varphi^{-1}(t) \mid \Psi(t)F(t,0) \mid\in L^p([t_0,\infty))$;
4). $\mid \Psi(t)\Psi^{-1}(g(t)) \mid\leq 1$, *for all* $t \geq t_0$;
5). The function $\omega = \omega(t)$ *given by*

$$\omega(t) = \left(\int_{t_0}^{t}[\varphi(s)\left|\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right|]^q ds\right)^{\frac{1}{q}} +$$

$$+\left(\int_{t}^{\infty}[\varphi(s)\left|\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right|]^q ds\right)^{\frac{1}{q}}$$

belongs to $L^r([t_0,\infty))$, $1 \leq r < \infty$.
Then, corresponding to each $\Psi-$ *bounded solution* $Z_0(t)$ *of equation (3) for which* $\Psi(t)Z_0(t)$ *belongs to* $L^r([t_0,\infty))$, *there exists a unique* $\Psi-$ *bounded solution* $Z(t)$ *of equation (8) for which* $\Psi(t)Z(t)$ *belongs to* $L^r([t_0,\infty))$ *and*

$$\lim_{t\to\infty} |\Psi(t)\left(Z(t) - Z_0(t)\right)| = 0. \tag{9}$$

Conversely, to each $\Psi-$ *bounded solution* $Z(t)$ *of equation (8) for which* $\Psi(t)Z(t)$ *belongs to* $L^r([t_0,\infty))$, *there corresponds a unique* $\Psi-$ *bounded solution* $Z_0(t)$ *of equation (3) for which* $\Psi(t)Z_0(t)$ *belongs to* $L^r([t_0,\infty))$ *such that (9) holds.*

Proof. We prove this theorem by means of the above Theorem 2.1.
Let $t_0 \geq 0$ be such that $\int_{t_0}^{\infty} (\lambda(t))^p \, dt \leq 2^{-p} K^{-p/q}$.
Consider the space

$$C_\Psi = \{Z : \mathbb{R}_+ \to \mathbb{M}_{d\times d} \mid Z \text{ is continuous and } \Psi- \text{ bounded on } [t_0, \infty)\}.$$

C_Ψ is a Banach space with respect to the norm $\mid Z \mid_\Psi = \sup\limits_{t\geq t_0} \mid \Psi(t)Z(t) \mid$.
For a fixed $\Psi-$ bounded solution $Z_0(t)$ of equation (3), define the operator $T : C_\Psi \to C_\Psi$, by

$$(TZ)(t) = Z_0(t) + \int_{t_0}^{t} X(t)P_1X^{-1}(s)F(s, Z(g(s)))ds -$$

$$- \int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s, Z(g(s)))ds, \ t \geq t_0.$$

From hypotheses, TZ exists and is continuous differentiable on $[t_0, \infty)$.
Indeed. for $v \geq t \geq t_0$,
$\mid \Psi(t) \int_t^v X(t)P_2X^{-1}(s)F(s, Z(g(s)))ds \mid=$
$=\mid \int_t^v \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\Psi(s)F(s, Z(g(s)))ds \mid\leq$
$\leq \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)F(s,Z(g(s)))|}{\varphi(s)} ds =$
$= \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)+F(s,0)]|}{\varphi(s)} ds \leq$
$\leq \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)]|+|\Psi(s)F(s,0)|}{\varphi(s)} ds \leq$
$\leq \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)]|}{\varphi(s)} ds+$
$+ \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)F(s,0)|}{\varphi(s)} ds \leq$
$\leq \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \lambda(s)h(\mid \Psi(s)Z(g(s)) \mid)ds+$
$+ \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)F(s,0)|}{\varphi(s)} ds \leq$
$\leq \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \lambda(s)h(\mid Z \mid_\Psi)ds+$
$+ \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)F(s,0)|}{\varphi(s)} ds \leq$
$\leq h(\mid Z \mid_\Psi) \cdot \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \lambda(s)ds+$
$+ \int_t^v \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)F(s,0)|}{\varphi(s)} ds \leq$
$\leq h(\mid Z \mid_\Psi) \cdot \left[\int_t^v \left(\varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\right)^q ds\right]^{1/q} \cdot \left[\int_t^v (\lambda(s))^p \, ds\right]^{1/p} +$
$+\left[\int_t^v \left(\varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\right)^q ds\right]^{1/q} \cdot \left[\int_t^v \left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p} \leq$
$\leq h(\mid Z \mid_\Psi) \cdot K^{1/q} \cdot \left[\int_t^v (\lambda(s))^p \, ds\right]^{1/p} + K^{1/q} \cdot \left[\int_t^v \left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p}$.
(I used the estimate $h(\mid \Psi(t)Z(g(t)) \mid) = h(\mid \Psi(t)\Psi^{-1}(g(t))\Psi(g(t))Z(g(t)) \mid) \leq$
$\leq h(\mid \Psi(t)\Psi^{-1}(g(t)) \mid \cdot \mid \Psi(g(t))Z(g(t)) \mid) \leq h(\mid \Psi(g(t))Z(g(t)) \mid) \leq h(\mid Z \mid_\Psi)$).
From hypotheses, it follows that the integral

$$\int_t^\infty X(t)P_2X^{-1}(s)F(s, Z(g(s)))ds$$

is convergent for all $Z \in C_\Psi$ and $t \geq t_0$.
Now, from hypotheses, TZ exists and is continuous differentiable on $[t_0, \infty)$.
This operator T has the following properties:
a). T maps C_Ψ into itself (i.e, T is well defined).
Indeed, for $Z \in C_\Psi$ and $t \geq t_0$,
$\mid \Psi(t)(TZ)(t) \mid\leq\mid \Psi(t)Z_0(t) \mid +$
$+ \int_{t_0}^t \varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)+F(s,0)]|}{\varphi(s)} ds+$

$+ \int_t^\infty \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)+F(s,0)]|}{\varphi(s)} ds \leq$
$\leq | Z_0 |_\Psi + h(| Z |_\Psi) \cdot \int_{t_0}^t \varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid \lambda(s)ds +$
$+ \int_{t_0}^t \varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid \left(\varphi^{-1}(s) \mid \Psi(s)F(s,0) \mid\right) ds +$
$+ h(| Z |_\Psi) \cdot \int_t^\infty \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \lambda(s)ds +$
$+ \int_t^\infty \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \left(\varphi^{-1}(s) \mid \Psi(s)F(s,0) \mid\right) ds \leq$
$\leq | Z_0 |_\Psi +$
$+ h(| Z |_\Psi) \cdot \left[\int_{t_0}^t \left(\varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid\right)^q ds\right]^{1/q} \cdot \left[\int_{t_0}^t (\lambda(s))^p ds\right]^{1/p} +$
$+ h(| Z |_\Psi) \cdot \left[\int_t^\infty \left(\varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\right)^q ds\right]^{1/q} \cdot \left[\int_t^\infty (\lambda(s))^p ds\right]^{1/p} +$
$+ \left[\int_{t_0}^t \left(\varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid\right)^q ds\right]^{1/q} \cdot \left[\int_{t_0}^t \left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p} +$
$+ \left[\int_t^\infty \left(\varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\right)^q ds\right]^{1/q} \cdot \left[\int_t^\infty \left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p} \leq$
$\leq | Z_0 |_\Psi + h(| Z |_\Psi) \cdot 2K^{1/q} \cdot \left[\int_{t_0}^\infty (\lambda(s))^p ds\right]^{1/p} +$
$+ 2K^{1/q} \cdot \left[\int_{t_0}^\infty \left(\varphi^{-1}(s) \mid \Psi(s)F(s,0) \mid\right)^p ds\right]^{1/p} < +\infty.$
b). T is strict h-contraction.
Indeed, for $Z_1, Z_2 \in C_\Psi$ and $t \geq t_0$,
$| \Psi(t) \left((TZ_1)(t) - (TZ_2)(t)\right) | \leq$
$\leq \int_{t_0}^t \varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)F(s,Z_1(g(s)))-F(s,Z_2(g(s)))|}{\varphi(s)} ds +$
$+ \int_t^\infty \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \frac{|\Psi(s)F(s,Z_1(g(s)))-F(s,Z_2(g(s)))|}{\varphi(s)} ds \leq$
$\leq h(| Z_1 - Z_2 |_\Psi) \cdot \int_{t_0}^t \varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid \lambda(s)ds +$
$+ h(| Z_1 - Z_2 |_\Psi) \cdot \int_t^\infty \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \lambda(s)ds \leq$
$\leq 2 \cdot h(| Z_1 - Z_2 |_\Psi) \cdot K^{1/q} \cdot \left[\int_{t_0}^\infty (\lambda(s))^p ds\right]^{1/p} \leq$
$\leq h(| Z_1 - Z_2 |_\Psi).$
Thus, $| TZ_1 - TZ_2 |_\Psi \leq h(| Z_1 - Z_2 |_\Psi)$ for all $Z_1, Z_2 \in C_\Psi$.
From Theorem 2.1, we obtain that there exists a unique fix point of T in C_Ψ.
This unique fix point Z(t) of T is solution of equation (8).
Indeed, for $t \geq t_0$, we have
$Z'(t) = (TZ(t))' = Z_0'(t) +$
$+ \int_{t_0}^t X'(t)P_1X^{-1}(s)F(s,Z(g(s)))ds + X(t)P_1X^{-1}(t)F(t,Z(g(t))) -$
$- \int_t^\infty X'(t)P_2X^{-1}(s)F(s,Z(g(s)))ds + X(t)P_2X^{-1}(t)F(t,Z(g(t))) =$
$= A(t)Z_0(t) +$
$+ \int_{t_0}^t A(t)X(t)P_1X^{-1}(s)F(s,Z(g(s)))ds - \int_t^\infty A(t)X(t)P_2X^{-1}(s)F(s,Z(g(s)))ds +$
$+ X(t)\left(P_1 + P_2\right)X^{-1}(t)F(t,Z(g(t))) =$
$= A(t)\left(TZ(t)\right) + F(t,Z(g(t))) = A(t)Z(t) + F(t,Z(g(t))).$
Let Z(t) the unique solution of equation (8), correspondent to $\Psi-$ bounded solution $Z_0(t)$ of equation (3) for which $\Psi(t)Z_0(t)$ belongs to $L^r([t_0,\infty))$.
From hypotheses, for a given $\varepsilon > 0$, we can find $t_1 > t_0$ such that

$$\left[\int_{t_1}^\infty (\lambda(s))^p ds\right]^{1/p} < \frac{\varepsilon}{6K^{1/q}h(| Z |_\Psi)}$$

and

$$\left[\int_{t_1}^\infty \left(\varphi^{-1}(s) \mid \Psi(s)F(s,0) \mid\right)^p ds\right]^{1/p} < \frac{\varepsilon}{6K^{1/q}}.$$

In addition, from Lemma 2.1, we can choose $t_2 > t_1$ such that

$$| \Psi(t)X(t)P_1 | < \frac{\varepsilon}{3M}, \text{ for } t > t_2,$$

where $M = \int_{t_0}^{t_1} | X^{-1}(s)\Psi^{-1}(s) || \Psi(s)F(s, Z(g(s))) | ds$.
Then, for $t > t_2$, we have
$|\Psi(t)(Z(t) - Z_0(t))| = |\Psi(t)((TZ)(t) - Z_0(t))| \leq$
$\leq \int_{t_0}^{t_1} | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) || \Psi(s)F(s, Z(g(s))) | ds+$
$+ \int_{t_1}^{t} \varphi(s) | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) | \varphi^{-1}(s) | \Psi(s)F(s, Z(g(s))) | ds+$
$+ \int_{t}^{\infty} \varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) | \varphi^{-1}(s) | \Psi(s)F(s, Z(g(s))) | ds.$
We now give an estimate for these three integrals:
$I_1 = \int_{t_0}^{t_1} | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) | \cdot | \Psi(s)F(s, Z(g(s))) | ds \leq$
$\leq | \Psi(t)X(t)P_1 | \cdot \int_{t_0}^{t_1} | X^{-1}(s)\Psi^{-1}(s) | \cdot | \Psi(s)F(s, Z(g(s))) | ds \leq$
$\leq \frac{\varepsilon}{3M} \cdot M = \frac{\varepsilon}{3}$, for $t > t_2$.
$I_2 = \int_{t_1}^{t} \varphi(s) | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) | \varphi^{-1}(s) | \Psi(s)F(s, Z(g(s))) | ds =$
$= \int_{t_1}^{t} \varphi(s) | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) | \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)+F(s,0)]|}{\varphi(s)} ds \leq$
$\leq h(| Z |_\Psi) \cdot \int_{t_1}^{t} \varphi(s) | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) | \lambda(s)ds+$
$+ \int_{t_1}^{t} \varphi(s) | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) | \left(\varphi^{-1}(s) | \Psi(s)F(s, 0) |\right) ds \leq$
$\leq h(| Z |_\Psi) \cdot \left[\int_{t_1}^{t} \left(\varphi(s) | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) |\right)^q ds\right]^{1/q} \cdot \left[\int_{t_1}^{t} (\lambda(s))^p ds\right]^{1/p} +$
$+ \left[\int_{t_1}^{t} \left(\varphi(s) | \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) |\right)^q ds\right]^{1/q} \cdot \left[\int_{t_1}^{t} \left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p} \leq$
$\leq h(| Z |_\Psi) \cdot K^{1/q} \cdot \left[\int_{t_1}^{t} (\lambda(s))^p ds\right]^{1/p} + K^{1/q} \cdot \left[\int_{t_1}^{t} \left(\varphi^{-1}(s) | \Psi(s)F(s, 0) |\right)^p ds\right]^{1/p} \leq$
$\leq h(| Z |_\Psi) \cdot K^{1/q} \cdot \frac{\varepsilon}{6K^{1/q}h(|Z|_\Psi)} + K^{1/q} \cdot \frac{\varepsilon}{6K^{1/q}} = \frac{\varepsilon}{3}$, for $t > t_2$.
$I_3 = \int_{t}^{\infty} \varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) | \varphi^{-1}(s) | \Psi(s)F(s, Z(g(s))) | ds =$
$= \int_{t}^{\infty} \varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) | \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)+F(s,0)]|}{\varphi(s)} ds \leq$
$\leq h(| Z |_\Psi) \cdot \int_{t}^{\infty} \varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) | \lambda(s)ds+$
$+ \int_{t}^{\infty} \varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) | \left(\varphi^{-1}(s) | \Psi(s)F(s, 0) |\right) ds \leq$
$\leq h(| Z |_\Psi) \cdot \left[\int_{t}^{\infty} \left(\varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) |\right)^q ds\right]^{1/q} \cdot \left[\int_{t}^{\infty} (\lambda(s))^p ds\right]^{1/p} +$
$+ \left[\int_{t}^{\infty} \left(\varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) |\right)^q ds\right]^{1/q} \cdot \left[\int_{t}^{\infty} \left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p} \leq$
$\leq h(| Z |_\Psi) \cdot K^{1/q} \cdot \left[\int_{t}^{\infty} (\lambda(s))^p ds\right]^{1/p} + K^{1/q} \cdot \left[\int_{t}^{\infty} \left(\varphi^{-1}(s) | \Psi(s)F(s, 0) |\right)^p ds\right]^{1/p} \leq$
$\leq h(| Z |_\Psi) \cdot K^{1/q} \cdot \frac{\varepsilon}{6K^{1/q}h(|Z|_\Psi)} + K^{1/q} \cdot \frac{\varepsilon}{6K^{1/q}} = \frac{\varepsilon}{3}$, for $t > t_2$.
It follows that

$$|\Psi(t)(Z(t) - Z_0(t))| \leq \varepsilon, \text{ for } t > t_2.$$

This shows that

$$\lim_{t\to\infty} |\Psi(t)(Z(t) - Z_0(t))| = 0$$

and (9) holds.
To prove the next statement of the theorem, consider a $\Psi-$ bounded solution $Z(t)$ of (8). Define

$$Z_0(t) = Z(t) - \int_{t_0}^{t} X(t)P_1X^{-1}(s)F(s, Z(g(s)))ds+$$

$$+ \int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s, Z(g(s)))ds, \ t \geq t_0.$$

With the previous arguments, one can show that $Z_0(t)$ is a unique $\Psi-$ bounded solution of (3) such that (9) holds.

The remainder of the proof demonstrates that $\Psi(t)Z(t)$ belongs to $L^r([t_0,\infty))$ provided $\Psi(t)Z_0(t)$ belongs to $L^r([t_0,\infty))$ and conversely.
The fix point of T gives the correspondence between the solutions $Z_0(t)$ and $Z(t)$:

$$Z(t) = Z_0(t) + \int_{t_0}^{t} X(t)P_1X^{-1}(s)F(s,Z(g(s)))ds-$$

$$-\int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s,Z(g(s)))ds,\ t\geq t_0.$$

From this and Hölder's inequality, we obtain (for $t \geq t_0$) :
$| \Psi(t)\,(Z(t)-Z_0(t))\,| \leq$
$\leq \int_{t_0}^{t}\varphi(s)\mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\mid \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)+F(s,0)]|}{\varphi(s)}ds+$
$+\int_{t}^{\infty}\varphi(s)\mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\mid \frac{|\Psi(s)[F(s,Z(g(s)))-F(s,0)+F(s,0)]|}{\varphi(s)}ds \leq$
$\leq h(|\, Z\,|_{\Psi})\cdot\int_{t_0}^{t}\varphi(s)\mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\mid \lambda(s)ds+$
$+\int_{t_0}^{t}\varphi(s)\mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\mid \left(\varphi^{-1}(s)\mid \Psi(s)F(s,0)\mid\right)ds+$
$+h(|\, Z\,|_{\Psi})\cdot\int_{t}^{\infty}\varphi(s)\mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\mid \lambda(s)ds+$
$+\int_{t}^{\infty}\varphi(s)\mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\mid \left(\varphi^{-1}(s)\mid \Psi(s)F(s,0)\mid\right)ds \leq$
$\leq h(|\, Z\,|_{\Psi})\cdot\left[\int_{t_0}^{t}\left(\varphi(s)\mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\mid\right)^q ds\right]^{1/q}\cdot\left[\int_{t_0}^{t}(\lambda(s))^p\,ds\right]^{1/p}+$
$+\left[\int_{t_0}^{t}\left(\varphi(s)\mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\mid\right)^q ds\right]^{1/q}\cdot\left[\int_{t_0}^{t}\left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p}+$
$+h(|\, Z\,|_{\Psi})\cdot\left[\int_{t}^{\infty}\left(\varphi(s)\mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\mid\right)^q ds\right]^{1/q}\cdot\left[\int_{t}^{\infty}(\lambda(s))^p\,ds\right]^{1/p}+$
$+\left[\int_{t}^{\infty}\left(\varphi(s)\mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\mid\right)^q ds\right]^{1/q}\cdot\left[\int_{t}^{\infty}\left(\frac{|\Psi(s)F(s,0)|}{\varphi(s)}\right)^p ds\right]^{1/p}\leq$
$\leq H\cdot\omega(t),$
where $H = h(|\, Z\,|_{\Psi})\cdot\left[\int_{t_0}^{\infty}(\lambda(s))^p\,ds\right]^{1/p}+\left[\int_{t_0}^{\infty}\left(\varphi^{-1}(s)\mid \Psi(s)F(s,0)\mid\right)^p ds\right]^{1/p}$ is a constant, for a fixed $Z(t)$.
It follows that

$$|\, \Psi(t)Z(t)\,| \leq |\, \Psi(t)Z_0(t)\,| + H\cdot\omega(t),\ \text{ for } t\geq t_0.$$

Since $\Psi(t)Z_0(t)$ and $H\cdot\omega(t)$ belong to $L^r([t_0,\infty))$, the function $\Psi(t)Z(t)$ belongs to $L^r([t_0,\infty))$. The converse follows similarly. ■

Remark 3.1 *If we put*

$$Z(t)=\begin{pmatrix} z_1 & z_1 & \cdots & z_1\\ z_2 & z_2 & \cdots & z_2\\ \vdots & \vdots & \vdots & \vdots\\ z_d & z_d & \cdots & z_d\end{pmatrix},\ F(t,Z)=\begin{pmatrix} f_1(t,z) & f_1(t,z) & \cdots & f_1(t,z)\\ f_2(t,z) & f_2(t,z) & \cdots & f_2(t,z)\\ \vdots & \vdots & \vdots & \vdots\\ f_d(t,z) & f_d(t,z) & \cdots & f_d(t,z)\end{pmatrix},$$

and $\Psi = I_d$, we obtain Theorem 2.1 from [11]. Thus, Theorem 3.1 generalizes and extends Theorem 2.1 from [11] in three directions: from systems of differential equations with modified argument to matrix differential equations with modified argument, the introduction of the matrix function Ψ that permits to obtaining a mixed asymptotic behavior for the components of solutions of the above equations and in connection with the belonging to space L^r of solutions.

The object of the next Theorem is to obtain new results in connection with $\Psi-$ asymptotic equivalence of two Lyapunov matrix differential equations with modified argument, namely (1) and (2).

Theorem 3.2 *Suppose that:*
1). There exist supplementary projections $P_1, P_2 \in \mathbb{M}_{d\times d}$, *a continuous function* $\varphi : \mathbb{R}_+ \to (0,\infty)$ *that satisfies the condition* $\int_0^\infty (\varphi(s))^q ds = +\infty$, $q > 1$, *and a constant* $K > 0$ *such the fundamental matrices* $X(t)$ *and* $Y(t)$ *for the linear matrix differential equations (3) and (4) respectively satisfy the condition*

$$\int_{t_0}^{t} [\varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right) \right|]^q ds +$$

$$+ \int_{t}^{\infty} [\varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right) \right|]^q ds \le K,$$

for all $t \ge t_0 \ge 0$, *where* t_0 *is sufficiently large;*
2). There exist a strict comparison function $h : \mathbb{R}_+ \to \mathbb{R}_+$ *and a function* $\lambda(t) \in L^p([t_0,\infty))$, $p > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *such that*

$$\varphi^{-1}(t) \mid \Psi(t)\,(F(t,Z_1) - F(t,Z_2)) \mid \le \lambda(t) h\left(\frac{1}{d} \mid \Psi(t)\,(Z_1 - Z_2) \mid\right)$$

for all $t \ge t_0$, $Z_1, Z_2 \in \mathbb{M}_{d\times d}$;
3). $\varphi^{-1}(t) \mid \Psi(t)F(t,0) \mid \in L^p([t_0,\infty))$;
4). $\mid \Psi(t)\Psi^{-1}(g(t)) \mid \le 1$, *for all* $t \ge t_0$;
5). The function $\mu = \mu(t)$ *given by*

$$\mu(t) = \left(\int_{t_0}^{t} [\varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right) \right|]^q ds\right)^{\frac{1}{q}} +$$

$$+ \left(\int_{t}^{\infty} [\varphi(s) \left| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right) \right|]^q ds\right)^{\frac{1}{q}}$$

belongs to $L^r([t_0,\infty))$, $1 \le r < \infty$.
Then, corresponding to each $\Psi-$ *bounded solution* $Z_0(t)$ *of equation (1) for which* $\Psi(t)Z_0(t)$ *belongs to* $L^r([t_0,\infty))$, *there exists a unique* $\Psi-$ *bounded solution* $Z(t)$ *of equation (2) for which* $\Psi(t)Z(t)$ *belongs to* $L^r([t_0,\infty))$ *and*

$$\lim_{t\to\infty} |\Psi(t)\,(Z(t) - Z_0(t))| = 0. \tag{10}$$

Conversely, to each $\Psi-$ *bounded solution* $Z(t)$ *of equation (2) for which* $\Psi(t)Z(t)$ *belongs to* $L^r([t_0,\infty))$, *there exists a unique* $\Psi-$ *bounded solution* $Z_0(t)$ *of equation (1) for which* $\Psi(t)Z_0(t)$ *belongs to* $L^r([t_0,\infty))$ *such that (10) holds.*

Proof. We will use Theorem 3.1, variant for systems of differential equations and some results from [7].
From Lemmas 2.4 and 2.6, [7], we know that $Z(t)$ is a $\Psi-$ bounded solution on $\mathbb{R}_+$ of equation (2) iff $z(t) = \mathcal{V}ec(Z(t))$ is a $I_d\otimes\Psi-$ bounded solution of the corresponding Kronecker product system associated with (2), i.e. system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z + f(t,z),\; t \ge t_0, \tag{11}$$

where $f(t,z) = Vec\,(F(t,Z))$, on the same interval $\mathbb{R}_+$.
We verify the hypotheses of Theorem 3.1, variant for systems of differential equations.
a). From Lemma 2.7, [7], we know that $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the homogeneous system associated with (11), i.e. system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z. \tag{12}$$

With the help of Lemma 2.1, [7], we have
$[\varphi(s) \parallel (I_d \otimes \Psi(t))\, U(t)\, (I_d \otimes P_i)\, U^{-1}(s)\, (I_d \otimes \Psi(s))^{-1} \parallel]^q =$
$= [\varphi(s) \left|\left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_iX^{-1}(s)\Psi^{-1}(s)\right)\right|]^q,$
for $t, s \in \mathbb{R}_+$ and $i = 1, 2$.
It follows that the hypothesis 1) of Theorem 3.1 is satisfied with $I_d \otimes \Psi$ in role of Ψ, $Y^T(t) \otimes X(t)$ in role of $X(t)$ and $I_d \otimes P_i$ in role of P_i.
b). With the help of Lemma 2.5, [7], we have, for $t \geq t_0$ and $z_1, z_2 \in \mathbb{R}^{d^2}$,
$\varphi^{-1}(t) \parallel (I_d \otimes \Psi(t))\, (f(t, z_1) - f(t, z_2)) \parallel=$
$= \varphi^{-1}(t) \parallel (I_d \otimes \Psi(t))\, (Vec\, (F(t, Z_1)) - Vec\, (F(t, Z_2))) \parallel=$
$= \varphi^{-1}(t) \parallel (I_d \otimes \Psi(t))\, Vec\, (F(t, Z_1) - F(t, Z_2)) \parallel\leq$
$\leq \varphi^{-1}(t) \mid \Psi(t)\, (F(t, Z_1) - F(t, Z_2)) \mid\leq$
$\leq \lambda(t)h\left(\frac{1}{d} \mid \Psi(t)\, (Z_1 - Z_2) \mid\right) \leq$
$\leq \lambda(t)h\, (\parallel (I_d \otimes \Psi(t))\, \mathcal{V}ec\, (Z_1 - Z_2) \parallel) =$
$= \lambda(t)h\, (\parallel (I_d \otimes \Psi(t))\, (z_1 - z_2) \parallel)$.
Consequently, the hypothesis 2) of Theorem 3.1 is satisfied with $I_d \otimes \Psi$ in role of Ψ and f in role of F.
c). With the help of Lemma 2.5, [7], we have, for $t \geq t_0$,
$\varphi^{-1}(t) \parallel (I_d \otimes \Psi(t))\, f(t, 0) \parallel= \varphi^{-1}(t) \parallel (I_d \otimes \Psi(t))\, \mathcal{V}ec\, (F(t, 0)) \parallel\leq$
$\leq \varphi^{-1}(t) \mid \Psi(t)F(t, 0) \mid\in L^p([t_0, \infty))$.
It follows that the hypothesis 3) of Theorem 3.1 is satisfied with $I_d \otimes \Psi$ in role of Ψ and f in role of F.
d). With the help of Lemmas 2.1 and 2.7, [7], we have, for $t \geq t_0$,
$\mu(t) = \left(\int_{t_0}^{t} [\varphi(s) \left|\left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right)\right|]^q ds\right)^{\frac{1}{q}} +$
$+ \left(\int_{t}^{\infty} [\varphi(s) \left|\left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right)\right|]^q ds\right)^{\frac{1}{q}} =$
$= \left(\int_{t_0}^{t} [\varphi(s) \left|(I_d \otimes \Psi(t))\, U(t)\, (I_d \otimes P_1)\, U^{-1}(s)\, (I_d \otimes \Psi(s))^{-1}\right|]^q ds\right)^{\frac{1}{q}} +$
$+ \left(\int_{t}^{\infty} [\varphi(s) \left|(I_d \otimes \Psi(t))\, U(t)\, (I_d \otimes P_2)\, U^{-1}(s)\, (I_d \otimes \Psi(s))^{-1}\right|]^q ds\right)^{\frac{1}{q}} =$
$= \overline{\omega(t)}$
Consequently, the hypothesis 5) of Theorem 3.1 is satisfied with $I_d \otimes \Psi$ in role of Ψ, $Y^T(t) \otimes X(t)$ in role of $X(t)$ and $I_d \otimes P_i$ in role of P_i.
We observe that the hypotheses of Theorem 3.1 are satisfied for in case of (11), in particular case $U(t)$, $I_d \otimes \Psi$ and $I_d \otimes P_i$ respectively.
Now, we finish the proof.
Let $Z_0(t)$ be a $\Psi-$ bounded solution of equation (1) for which $\Psi(t)Z_0(t)$ belongs to $L^r([t_0, \infty))$. From Lemmas 2.5 and 2.6, [7], the function $z_0(t) = \mathcal{V}ec(Z_0(t))$ is a $I_d \otimes \Psi-$ bounded solution of equation (12) and belongs to $L^r([t_0, \infty))$. From Theorem 3.1, variant for systems, there exists a unique $I_d \otimes \Psi-$ bounded solution $z(t)$ of equation (11) for which $(I_d \otimes \Psi(t))\, z$ belongs to $L^r([t_0, \infty))$ such that

$$\lim_{t\to\infty} |(I_d \otimes \Psi(t))\, (z(t) - z_0(t))| = 0. \tag{13}$$

From Lemmas 2.4, 2.5 and 2.6, [7], we obtain that $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is a $\Psi-$ bounded solution of equation (2) that satisfies (10) and belongs to $L^r([t_0, \infty))$.
For the last statement of Theorem, let Z(t) be a $\Psi-$ bounded solution of equation (2) for which $\Psi(t)Z(t)$ belongs to $L^r([t_0, \infty))$. From Lemmas 2.4, 2.5 and 2.6, [7], the function $z(t) = \mathcal{V}ec(Z(t))$ is a $I_d \otimes \Psi-$ bounded solution of equation (11) for which $(I_d \otimes \Psi(t))\, z$ belongs to $L^r([t_0, \infty))$. From Theorem 3.1, variant for systems, there exists a unique $I_d \otimes \Psi-$ bounded solution $z_0(t)$ of equation (12) for which $(I_d \otimes \Psi(t))\, z_0$ belongs to $L^r([t_0, \infty))$ and (13) holds.

From Lemmas 2.4, 2.5 and 2.6, [7], we obtain that $Z_0(t) = \mathcal{V}ec^{-1}(z_0(t))$ is a $\Psi-$ bounded solution of equation (1) that satisfies (10) and belongs to $L^r([t_0, \infty))$. ■

Remark 3.2 *The introduction of the function φ does not alter conditions (9) and (10) of the theorems; this function φ can only serve to weaken the required hypotheses on F.*
Instead, the introduction of matrix Ψ allows to obtain mixed asymptotic behaviors for the components of the solutions of matrix differential equations.

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