On the $\Psi-$uniform asymptotic stability of nonlinear Lyapunov matrix differential equations

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Abstract. This paper deals with obtaining (necessary and) sufficient conditions for $\Psi-$ uniform asymptotic stability of solutions of nonlinear Lyapunov matrix differential equations.

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1 Introduction

The Lyapunov matrix differential equations occur in many branches of applied matematics.

The purpose of this paper is to provide sufficient conditions for $\Psi-$uniform asymptotic stability of trivial solution of nonlinear Lyapunov matrix differential equations of the form

$$Z' = A(t)Z + F(t, Z) \tag{1}$$

$$Z' = A(t)Z + ZB(t) + F(t, Z) \tag{2}$$

$$Z' = A(t)Z + ZB(t) + \int_0^t G(t, s, Z(s))ds \tag{3}$$

as a perturbed equations of linear matrix differential equations

$$Z' = A(t)Z, \tag{4}$$

$$Z' = ZB(t), \tag{5}$$

$$Z' = A(t)Z + ZB(t). \tag{6}$$

We investigate conditions on the fundamental matrices of the equations (4), (5) and (6) and on the coefficients of equations under which the trivial solutions of the equations (1) - (6) are $\Psi-$uniformly asymptotically stable on $\mathbb{R}_+$.

Here, Ψ is a matrix function whose introduction permits us obtaining a mixed asymptotic behavior for the components of solutions.

Recent results for $\Psi-$boundedness, $\Psi-$stability, $\Psi-$instability, $\Psi-$conditional asymptotic stability, dichotomy and conditioning for Lyapunov matrix differential equations or systems

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of differential equations have been given in many papers. See, for example, [2], [5], [6], [7], [9], [10], [11], [12], [13] and in the references there.

The results obtained in this work generalize results from works [3], [4], [5], [7].

The main tools used in this paper are the variation of constants formula, Gronwall's inequality and the technique of Kronecker product of matrices, which have been successfully applied in various fields of matrix theory.

2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let R^n be the Euclidean $n-$ dimensional space. For $x = (x_1, x_2, ..., x_n)^T \in R^n$, let $\| x \| = \max\{| x_1|, | x_2|, ..., | x_n|\}$ be the norm of x (here, ^{T} denotes transpose).

Let $\mathbb{M}_{n\times n}$ be the linear space of all real $n \times n$ matrices.

For $A = (a_{ij}) \in \mathbb{M}_{n\times n}$, we define the norm $| A |$ by formula $| A | = \sup\limits_{\|x\|\leq 1} \| Ax \|$. It is well-known that $| A | = \max\limits_{1\leq i\leq n}\{\sum\limits_{j=1}^{n} | a_{ij}|\}$.

By a solution of the equation (1) (or (2) - (6)) we mean a continuous differentiable $n \times n$ matrix function $Z(t)$ satisfying the equation (1) (or (2) - (6) respectively) for all $t \in R_+ = [0, \infty)$.

In equations (4) - (6), we assume that A and B are continuous $n\times n$ matrix functions on R_+. It is well-known that these conditions ensure the existence and uniqueness of the solutions of these equations passing through any given point $(t_0, Z_0) \in R_+ \times \mathbb{M}_{n\times n}$ and is defined on R_+.

In addition, for equations (1) - (3), we assume that $F : R_+ \times \mathbb{M}_{n\times n} \longrightarrow \mathbb{M}_{n\times n}$ is continuous matrix function such that $F(t, O_n) = O_n$ and $G : D \longrightarrow \mathbb{M}_{n\times n}$, $D = \{(t, s, Z) \mid 0 \leq s \leq t < +\infty, Z \in \mathbb{M}_{n\times n}\}$, is continuous matrix function such that $G(t, s, O_n) = O_n$. It is well-known that these conditions ensure the local existence of a solution of these equations passing through any given point $(t_0, Z_0) \in R_+ \times \mathbb{M}_{n\times n}$, but it does not guarantee that the solution is unique or that it can be continued for large values of $t \in R_+$.

Let $\Psi_i : \mathrm{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., n$, be continuous functions and

$$\Psi = \text{diag}\,[\Psi_1, \Psi_2, \cdots \Psi_n].$$

In this paper, we assume a natural hypotheses in studying $\Psi-$ stability of trivial solution of (1) - (3). See, for example [5] and [7] (here this hypothesis is tacitly used in particular case $\Psi = I_n$).

(H_1) For all $t_0 \in R_+$ and $Z_0 \in \mathbb{M}_{n\times n}$, there exists a unique solution $Z(t)$ of the equation (1) on an interval $[t_0, t_+)$, $t_0 < t_+ \leq +\infty$, such that $Z(t_0) = Z_0$.

(H_2) For all $t_0 \in R_+$ and $Z_0 \in \mathbb{M}_{n\times n}$, there exists a unique solution $Z(t)$ of the equation (2) on an interval $[t_0, t_+)$, $t_0 < t_+ \leq +\infty$, such that $Z(t_0) = Z_0$.

(H_3) For all $t_0 > 0$, $Z_0 \in \mathbb{M}_{n\times n}$ and $\rho > 0$, if $| \Psi(t_0)Z_0 | < \rho$, there exists a unique solution $Z(t)$ on R_+ of the equation (3) which satisfies the equality $Z(t_0) = Z_0$ and the inequality $| \Psi(t)Z(t) | \leq \rho$ for all $t \in [0, t_0]$.

Now, we recall the definitions of various types of $\Psi-$ stability that we need in what follows. In this Definition, the equation $Z' = F(t, Z)$ is a general matrix differential equation.

Definition 2.1 *(i) The trivial solution of the equation* $Z' = F(t, Z)$ *is said to be* $\Psi-$*stable over* R_+ *if for each* $\varepsilon > 0$ *and each* $t_0 \in R_+$, *there is a a corresponding* $\delta = \delta(\varepsilon, t_0) > 0$ *such that any solution* $Z(t)$ *of equation which satisfies the inequality* $\mid \Psi(t_0)Z(t_0) \mid < \delta$, *exists and satisfies the inequality* $\mid \Psi(t)Z(t) \mid < \varepsilon$ *for all* $t \geq t_0$.
(ii) The trivial solution of the equation $Z' = F(t, Z)$ *is said to be* $\Psi-$*uniformly stable over* R_+ *if it is* $\Psi-$*stable over* R_+ *and the above* δ *is independent of* t_0.
(iii). The trivial solution of the equation $Z' = F(t, Z)$ *is said to* $\Psi-$*uniformly asymptotically stable on* R_+ *if it is* $\Psi-$*uniformly stable over* R_+ *and in addition, there is a* $\delta_0 > 0$ *and, for each* $\varepsilon > 0$, *a corresponding* $T(\varepsilon) > 0$ *such that any solution* $Z(t)$ *of equation which satisfies the inequality* $\mid \Psi(t_0)Z(t_0) \mid < \delta_0$ *for some* $t_0 \geq 0$, *satisfies the inequality* $\mid \Psi(t)Z(t) \mid < \varepsilon$ *for all* $t \geq t_0 + T(\varepsilon)$.
(iv). The trivial solution of the equation $Z' = F(t, Z)$ *is said to* $\Psi-$*exponentially asymptotically stable on* R_+ *if there exists* $\lambda > 0$ *and, for any* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that any solution* $Z(t)$ *of equation which satisfies the inequality* $\mid \Psi(t_0)Z(t_0) \mid < \delta(\varepsilon)$ *for some* $t_0 \geq 0$, *satisfies the inequality* $\mid \Psi(t)Z(t) \mid < \varepsilon e^{-\lambda(t-t_0)}$ *for all* $t \geq t_0$.

Remark 2.1 *1. Our Definition extends the definition of (uniform asymptotic, exponential) stability from (vector) differential equations to matrix differential equations.*
2. For $\Psi = I_d$, *one obtain the notion of classical stability (see [3], [4]).*
3. It is easy to see that if Ψ *and* Ψ^{-1} *are bounded on* R_+, *then the* $\Psi-$ *stability is equivalent with the classical stability.*

We now give some definitions and properties in connection with Kronecker product of matrices and vectorization operator.

Definition 2.2 *([1]) Let* $A = (a_{ij}) \in M_{m\times n}$ *and* $B = (b_{ij}) \in M_{p\times q}$. *The Kronecker product of A and B, written* $A \otimes B$, *is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

Lemma 2.1 *([1]) The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions exist:*

1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
2). $(A \otimes B)^T = A^T \otimes B^T$;
3). $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$;
4). $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
5). $A \otimes (B + C) = A \otimes B + A \otimes C$;
6). $(A + B) \otimes C = A \otimes C + B \otimes C$;
7). $I_n \otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;
8).$(A(t) \otimes B(t))' = A'(t) \otimes B(t) + A(t) \otimes B'(t)$; *(′ denotes the derivative* $\frac{d}{dt}$*).*

Proof. See in [1]. ■

Definition 2.3 *([8]) The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called the vectorization operator.

Lemma 2.2 *The vectorization operator*

$$\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn},\ A \longrightarrow \mathcal{V}ec(A),$$

is a linear and one-to-one operator. In addition, $\mathcal{V}ec$ and $\mathcal{V}ec^{-1}$ are continuous operators.

Proof. See Lemma 2, [6]. ■

Remark 2.2 *Obviously, a function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is a continuous (differentiable) matrix function on R_+ if and only if the function $f : R_+ \longrightarrow \mathbb{R}^{n^2}$, defined by $f(t) = \mathcal{V}ec(F(t))$, is a continuous (differentiable) vector function on R_+.*

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations.

Lemma 2.3 *([8]) If $A, B, M \in \mathbb{M}_{n\times n}$, then*

1). $\mathcal{V}ec(AMB) = (B^T \otimes A)\cdot \mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^T \otimes I_n)\cdot \mathcal{V}ec(M)$;
3). $\mathcal{V}ec(AM) = (I_n \otimes A)\cdot \mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^T \otimes A)\cdot \mathcal{V}ec(I_n)$.

Proof. See [8], Chapter 2. ■

The Lemmas which follows are one of the most useful technical results in the proofs of main results of present paper.

Lemma 2.4 *([6]) The matrix function $Z(t)$ is a solution on R_+ of (2) if and only if the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the differential system*

$$z' = \left(I_n \otimes A(t) + B^T(t) \otimes I_n\right) z + f(t,z), \tag{7}$$

where $f(t,z) = \mathcal{V}ec\,(F(t,Z))$, on the same interval R_+.

Proof. See Lemma 5, [6]. ■

Definition 2.4 *([6]) The above system (7) is called "corresponding Kronecker product system associated with (2)".*

Lemma 2.5 *([6]). For every matrix function $M : R_+ \longrightarrow \mathbb{M}_{n\times n}$,*

$$\frac{1}{n} \mid \Psi(t)M(t) \mid \leq \parallel (I_n \otimes \Psi(t))\,\mathcal{V}ec\,(M(t)) \parallel_{\mathbb{R}^{n^2}} \leq \mid \Psi(t)M(t) \mid, \forall t \geq 0. \tag{8}$$

Proof. See Lemma 6, [6]. ■

Lemma 2.6 *([6]). The trivial solution of the equation (2) or (3) is $\Psi-$(uniformly, uniformly asymptotically) stable on R_+ if and only if the trivial solution of the corresponding Kronecker product system associated with (2) or (3) respectively, is $I_n \otimes \Psi-$(uniformly, uniformly asymptotically) stable on R_+.*

Proof. It results from Definitions of $\Psi-$stability and above Lemma 2.5. ■

Lemma 2.7 *([6]). Let X(t) and Y(t) be a fundamental matrices for the equations*

$$Z' = A(t)Z \tag{9}$$

$$Z' = ZB(t) \tag{10}$$

respectively.
Then, the matrix $Z(t) = Y^T(t) \otimes X(t)$ *is a fundamental matrix for the linear differential system*

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z \tag{11}$$

(i.e. for homogeneous differential system associated with (7)).

Proof. See Lemma 9, [6]. ■

At the end of this section, we recall a result which is useful in the proof of one of our main result.

Lemma 2.8 *(Theorem 4, [5]). Suppose that:*
1). for all $t_0 > 0$, $x_0 \in R^n$ *and* $\rho > 0$, *if* $\| \Psi(t_0)x_0 \| < \rho$, *there exists a unique solution x(t) on* R_+ *of the equation* $x' = A(t)x + \int_0^t f(t, s, x(s))ds$ *such that* $x(t_0) = x_0$ *and* $\| \Psi(t)x(t) \| \leq \rho$ *for all* $t \in [0, t_0]$;
2). the fundamental matrix X(t) of the equation $x' = A(t)x$ *satisfies the condition*

$$| \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) | \leq Ke^{-\lambda(t-s)}, \text{ for } 0 \leq s \leq t < +\infty,$$

where K and λ *are positive constants;*
3). the vector function $f(t, s, x)$ *satisfies the condition*

$$\| \Psi(t)f(t, s, x) \| \leq k(t - s) \| \Psi(t)x \|,$$

for $0 \leq s \leq t < \infty$ *and for all* $x \in R^n$, *where* $k : R_+ \to R_+$ *satisfies the conditions*

$$\int_0^\infty k(t)dt < \frac{\lambda}{K} \text{ and } \lim_{t\to\infty} k(t) = 0.$$

Then, the trivial solution of equation $x' = A(t)x + \int_0^t f(t, s, x(s))ds$ *is* $\Psi-$*uniformly asymptotically stable on* R_+.

3 Ψ−uniform asymptotic stability of the linear (Lyapunov) matrix differential equations

The purpose of this section is to provide (necessary and) sufficient conditions for $\Psi-$uniform asymptotic stability of linear (Lyapunov) matrix differential equations (4) − (6).

Theorem 3.1 *Let* $X(t)$ *be a fundamental matrix for equation (4).*
Then, the trivial solution of equation (4) is $\Psi-$*uniformly asymptotically stable on* R_+ *if and only if there exist the constants* $K > 0$ *and* $\alpha > 0$ *such that*

$$| \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) | \leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

Proof. The solution of equation (4) which takes the value $C_0 \in \mathbb{M}_{n\times n}$ at $t_0 \geq 0$ is

$$Z(t) = X(t)X^{-1}(t_0)C_0, \text{for } t \geq t_0 \geq 0.$$

Suppose first that the trivial solution of equation (4) is $\Psi-$uniformly asymptotically stable on R_+. From Definition 2.1, this solution is $\Psi-$uniformly stable on R_+ and there is a $\delta_0 > 0$, and for each $\varepsilon \in (0, \delta_0/2)$, there exists $T = T(\varepsilon) > 0$ such that any solution $Z(t)$ of (4) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid< \delta_0$ for some $t_0 \geq 0$, satisfies the inequality $\mid \Psi(t)Z(t) \mid< \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$, or

$$\mid \Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0)\Psi(t_0)C_0 \mid< \varepsilon, \text{ for all } t \geq t_0 + T(\varepsilon).$$

Let $C \in \mathbb{M}_{n\times n}$ be such that $\mid C \mid\leq 1$ and $C_0 = \frac{1}{2}\delta_0\Psi^{-1}(t_0)C$. Since $\mid \Psi(t_0)C_0 \mid< \delta_0$, it follows that

$$\mid \Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0)\frac{1}{2}\delta_0 C \mid< \varepsilon, \text{ for all } t \geq t_0 + T(\varepsilon).$$

Therefore,

$$\mid \Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0) \mid\leq \frac{2\varepsilon}{\delta_0} = \theta < 1, \text{ for all } t \geq t_0 + T(\varepsilon).$$

Then, for $t_0 = \tau \geq 0$ and $t = \tau + T(\varepsilon)$ we have

$$\mid \Psi(\tau + T(\varepsilon))X(\tau + T(\varepsilon))X^{-1}(\tau)\Psi^{-1}(\tau) \mid\leq \theta < 1.$$

Since the trivial solution of (4) is $\Psi-$uniformly stable on R_+, there exists a constant $M > 0$ such that

$$\mid \Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0) \mid\leq M, \text{ for all } 0 \leq t_0 \leq t < +\infty,$$

or, so much the more,

$$\mid \Psi(t+h)X(t+h)X^{-1}(t)\Psi^{-1}(t) \mid\leq M, \text{ for } t, h \in [0, T(\varepsilon)].$$

Now, for $t \geq s \geq 0$, there exists $n \in \mathbb{N}$ such that $s + nT(\varepsilon) \leq t < s + (n+1)T(\varepsilon)$. Then, we can write (to simplify the writing, we put T instead of $T(\varepsilon)$)

$$\mid \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) \mid=$$

$$=\mid \Psi(t)X(t)X^{-1}(s+T)\Psi^{-1}(s+T)\Psi(s+T)X(s+T)X^{-1}(s)\Psi^{-1}(s) \mid\leq$$

$$\leq\mid \Psi(t)X(t)X^{-1}(s+T)\Psi^{-1}(s+T) \mid \cdot \mid \Psi(s+T)X(s+T)X^{-1}(s)\Psi^{-1}(s) \mid\leq$$

$$\leq \theta \mid \Psi(t)X(t)X^{-1}(s+T)\Psi^{-1}(s+T) \mid\leq$$

$$\leq \theta^2 \mid \Psi(t)X(t)X^{-1}(s+2T)\Psi^{-1}(s+2T) \mid\leq \cdots$$

$$\leq \theta^n \mid \Psi(t)X(t)X^{-1}(s+nT)\Psi^{-1}(s+nT) \mid\leq M\theta^n = Me^{n\ln\theta}.$$

Since $n \leq \frac{t-s}{T} < n+1$, it follows that $n\ln\theta \geq \frac{t-s}{T}\ln\theta > (n+1)\ln\theta$ and then $n\ln\theta < \frac{t-s}{T}\ln\theta - \ln\theta$.
Thus, if we set $\alpha = -T^{-1}\ln\theta > 0$ and $K = \theta^{-1}M > 0$, we have

$$\mid \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) \mid\leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

Suppose next that for equation (4) there exist the constants $K > 0$ and $\alpha > 0$ such that

$$| \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) | \leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

From Definition 2.1 we see that the trivial solution of (4) is $\Psi-$uniformly stable on R_+. Now, for $\delta_0 = 1$ and $T(\varepsilon) = -\alpha^{-1} \ln \frac{\varepsilon}{K}$ (with $\varepsilon \in (0, K)$), for any solution Z(t) of (4) which satisfies the inequality $| \Psi(t_0)Z(t_0) | < \delta_0 = 1$, $t_0 \geq 0$, we have, for all $t \geq t_0 + T(\varepsilon)$,

$$| \Psi(t)Z(t) | =$$

$$= | \Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0)\Psi(t_0)Z(t_0) | \leq$$

$$\leq | \Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0) | \cdot | \Psi(t_0)Z(t_0) | \leq$$

$$\leq Ke^{-\alpha(t-t_0)} | \Psi(t_0)Z(t_0) | < \varepsilon.$$

Hence, the trivial solution of (4) is $\Psi-$uniformly asymptotically stable on R_+. ■

Remark 3.1 *1. In the same manner as in classical stability for systems of linear differential equations, we can speak about $\Psi-$uniform asymptotic stability of a linear matrix differential equation (4).*
2. In the same manner as in classical stability for systems of linear differential equations, it is easy to see that for linear matrix equation (4), $\Psi-$uniform asymptotic stability and $\Psi-$exponential stability are equivalent.

Theorem 3.1 contains as a particular case a result concerning $\Psi-$uniform asymptotic stability of systems of differential equations of the form

$$z' = A(t)z.$$

Indeed, consider in equation (4)

$$Z = \begin{pmatrix} z_1 & z_1 & \cdots & z_1 \\ z_2 & z_2 & \cdots & z_2 \\ \vdots & \vdots & \vdots & \vdots \\ z_n & z_n & \cdots & z_n \end{pmatrix},$$

where $z = (z_1, z_2, ...z_n)^T$.

Now, the definitions and conditions for $\Psi-$uniform asymptotic stability on R_+ for solutions of this system and (4) are the same. Thus, we have

Corollary 3.1 *(Theorem 1, [5]) Let X(t) be a fundamental matrix for system $z' = A(t)z$. Then, the trivial solution of the system above is $\Psi-$uniformly asymptotically stable on R_+ if and only if there exist the constants $K > 0$ and $\alpha > 0$ such that*

$$| \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) | \leq Ke^{-\alpha(t-s)}, \textit{ for } 0 \leq s \leq t < +\infty.$$

The next result shows that $\Psi-$uniform asymptotic stability of (4) is preserved under small or absolutely integrable perturbations of the coefficient matrix $A(t)$.

Proposition 3.1 *Suppose that:*
1). A(t) is a continuous matrix function on R_+ and the matrix differential equation (4) is $\Psi-$uniformly asymptotically stable on R_+.

2). $A_1(t)$ is a continuous matrix function on R_+ such that satisfies one of the following conditions:
i). $M = \sup_{t\geq 0} \mid \Psi(t)A_1(t)\Psi^{-1}(t) \mid$ is a sufficiently small number;
ii). $L = \int_0^\infty \mid \Psi(t)A_1(t)\Psi^{-1}(t) \mid dt < +\infty$;
iii). $\lim_{t\to\infty} \mid \Psi(t)A_1(t)\Psi^{-1}(t) \mid = 0$.
Then, the linear matrix differential equation

$$Z' = [A(t) + A_1(t)]Z$$

is $\Psi-$uniformly asymptotically stable on R_+.

Proof. It is similar with the proof of the Theorem 4.1 below, in particular case $F(t, Z) = A_1(t)Z$. ■

Remark 3.2 *This Proposition generalizes a similar result from [3] (Chapter 3, Theorem 1), in connection with classical uniform asymptotic stability of systems of linear differential equations of the form $z' = A(t)z$. The cases ii) and iii) extend the result from [4] (Chapter 1, Proposition 1).*

Corollary 3.2 *Suppose that:*
1). For matrix function Ψ, there exist the constants $K > 0$ and $\alpha > 0$ such that

$$\mid \Psi(t)\Psi^{-1}(s) \mid \leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

2). A(t) is a continuous matrix function on R_+ such that satisfies one of the following conditions:
i). $M = \sup_{t\geq 0} \mid \Psi(t)A(t)\Psi^{-1}(t) \mid$ is a sufficiently small number;
ii). $L = \int_0^\infty \mid \Psi(t)A(t)\Psi^{-1}(t) \mid dt < +\infty$;
iii). $\lim_{t\to\infty} \mid \Psi(t)A(t)\Psi^{-1}(t) \mid = 0$.
Then, the linear matrix differential equation (4) is $\Psi-$uniformly asymptotically stable on R_+.

Proof. Indeed, this follows from the Proposition 3.1. ■

The next result is similar.

Proposition 3.2 *Suppose that:*
1). A(t) is a continuous matrix function on R_+ and the matrix differential equation (4) is $\Psi-$uniformly asymptotically stable on R_+.
2). B(t) is a continuous matrix function on R_+ such that satisfies one of the following conditions:
i). $M = \sup_{t\geq 0} \mid B(t) \mid$ is a sufficiently small number;
ii). $L = \int_0^\infty \mid B(t) \mid dt < +\infty$;
iii). $\lim_{t\to\infty} \mid B(t) \mid = 0$.
Then, the linear matrix differential equation (6) is $\Psi-$uniformly asymptotically stable on R_+.

Proof. It is similar with the proof of the Theorem 4.1 below, in particular case $F(t, Z) = ZB(t)$. ■

Remark 3.3 *Proposition may be interpreted as saying that if the matrix differential equation (4) is $\Psi-$uniformly asymptotically stable on R_+, then the linear Lyapunov matrix differential equation (6) is $\Psi-$uniformly asymptotically stable on R_+.*

Corollary 3.3 *Suppose that:*
1). For matrix function Ψ*, there exist the constants* $K > 0$ *and* $\alpha > 0$ *such that*

$$| \Psi(t)\Psi^{-1}(s) | \leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

2). B(t) is a continuous matrix function on R_+ *such that satisfies one of the following conditions:*
i). $M = \sup\limits_{t\geq 0} | B(t) |$ *is a sufficiently small number;*
ii). $L = \int_0^\infty | B(t) | dt < +\infty$;
iii). $\lim\limits_{t\to\infty} | B(t) |= 0$.
Then, the linear matrix differential equation (5) is $\Psi-$*uniformly asymptotically stable on* R_+.

Proof. Indeed, this follows from the Proposition 3.2. ■

Theorem 3.2 *Let X(t) and Y(t) be a fundamental matrices for the equations (4) and (5) respectively.*
Then, the linear Lyapunov matrix differential equation (6) is $\Psi-$*uniformly asymptotically stable on* R_+ *if and only if there exist* $K > 0$ *and* $\alpha > 0$ *such that*

$$| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right) | \leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

Proof. From Lemma 2.6 one know that the equation (6) is $\Psi-$uniformly asymptotically stable on R_+ if and only if the corresponding Kronecker product system associated with equation (6), i.e. the system

$$z' = \left(I \otimes A(t) + B^T(t) \otimes I\right) z, \tag{12}$$

is $I \otimes \Psi-$uniformly asymptotically stable on R_+.
From Lemma 2.7 one know that the matrix $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the linear differential system (12).
A short computation shows that, for $t \geq s \geq 0$,

$$(I \otimes \Psi(t)) \left(U(t)U^{-1}(s)\right) (I \otimes \Psi(s))^{-1} =$$

$$= \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right).$$

Now, from Theorem 3.1, we have that the system (12) is $I \otimes \Psi-$uniformly asymptotically stable on R_+.
Once again, from Lemma 2.6, we have that the equation (6) is $\Psi-$uniformly asymptotically stable on R_+. ■

Corollary 3.4 *Let Y(t) be a fundamental matrix for the equation (5).*
Then, the linear Lyapunov matrix differential equation (5) is $\Psi-$*uniformly asymptotically stable on* R_+ *if and only if there exist* $K > 0$ *and* $\alpha > 0$ *such that*

$$| \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)\Psi^{-1}(s)\right) | \leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

Proof. Indeed, this follows from the Theorem 3.2. ■

Corollary 3.5 *Suppose that is satisfied one of the following conditions:*
i). the equation (4) is $\Psi-$*uniformly asymptotically stable on* R_+ *and the equation (5) is uniformly stable on* R_+;
ii). the equation (4) is $\Psi-$*uniformly stable on* R_+ *and the equation (5) is uniformly asymptotically stable on* R_+;
Then, the equation (6) is $\Psi-$*uniformly asymptotically stable on* R_+.

Proof. It results from the Theorem 3.2 and from inequality $| A \otimes B |\leq| A | \cdot | B |$, for all $A, B \in \mathbb{M}_{n\times n}$. ■

4 $\Psi-$uniform asymptotic stability of a nonlinear Lyapunov matrix differential equation

The purpose of this section is to provide sufficient conditions for $\Psi-$uniform asymptotic stability of trivial solution of nonlinear Lyapunov matrix differential equations (1) - (3).

Theorem 4.1 *Suppose that:*
1). The hypothesis (H_1) *is satisfied;*
2). The equation (4) is $\Psi-$*uniformly asymptotically stable on* R_+;
3). The matrix function $F : R_+ \times \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ *satisfies the inequality*

$$\mid \Psi(t)F(t,Z) \mid\leq \gamma(t) \mid \Psi(t)Z \mid$$

for all $t \in R_+$ *and* $Z \in \mathbb{M}_{n\times n}$, *where* $\gamma : R_+ \to R_+$ *is a continuous function that satisfies one of the following conditions:*
i). $M = \sup_{t\geq 0} \gamma(t)$ *is a sufficiently small number;*
ii). $L = \int_0^\infty \gamma(t)dt < +\infty$;
iii). $\lim_{t\to\infty} \gamma(t) = 0$.
Then, the trivial solution of the nonlinear matrix differential equation (1) is $\Psi-$*uniformly asymptotically stable on* R_+.

Proof. Let X(t) be a fundamental matrix for linear equation (4). From hypothesis 2) and Theorem 3.1, there exist $K > 0$ and $\alpha > 0$ such that

$$\mid \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) \mid\leq Ke^{-\alpha(t-s)},\ \text{for } 0 \leq s \leq t < +\infty.$$

If Z(t) is the solution of (1) with $Z(t_0) = Z_0$, by variation of constants formula (see [3], Ch II, s 2(8)),

$$Z(t) = X(t)X^{-1}(t_0)Z_0 + \int_{t_0}^{t} X(t)X^{-1}(s)F(s,Z(s))ds,\ \text{for } t \in [t_0,t_+).$$

From this,

$$\Psi(t)Z(t) = \left(\Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0)\right)(\Psi(t_0)Z_0) +$$

$$+\int_{t_0}^{t}\left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right)(\Psi(s)F(s,Z(s)))\,ds,\ \text{for } t \in [t_0,t_+),$$

and then, from hypothesis 3),

$$\mid \Psi(t)Z(t) \mid\leq Ke^{-\alpha(t-t_0)} \mid \Psi(t_0)Z_0 \mid +$$

$$+K\int_{t_0}^{t} e^{-\alpha(t-s)}\gamma(s) \mid \Psi(s)Z(s) \mid ds,\ \text{for } t \in [t_0,t_+).$$

Thus, the scalar function $w(t) = e^{\alpha(t-t_0)} \mid \Psi(t)Z(t) \mid$ satisfies the inequality

$$w(t) \leq Kw(t_0) + K\int_{t_0}^{t}\gamma(s)w(s)ds,\ \text{for } t \in [t_0,t_+).$$

By Gronwall's inequality (see [3], Ch I, Lemma 3), this implies

$$w(t) \leq Kw(t_0)e^{K\int_{t_0}^{t}\gamma(s)ds},\ \text{for } t \in [t_0,t_+). \tag{13}$$

In what follows, we have three cases:
In the case i), from (13) one obtain

$$w(t) \leq Kw(t_0)e^{KM(t-t_0)}, \text{ for } t \in [t_0, t_+)$$

and hence

$$|\, \Psi(t) Z(t) \,| \leq Ke^{-(\alpha-KM)(t-t_0)} \,|\, \Psi(t_0) Z_0 \,|, \text{ for } t \in [t_0, t_+).$$

In the case ii), from (13) one obtain

$$|\, \Psi(t) Z(t) \,| \leq Ke^{KL} e^{-\alpha(t-t_0)} \,|\, \Psi(t_0) Z_0 \,|, \text{ for } t \in [t_0, t_+).$$

In the case iii), there exist the constants $N > 0$ and $t^* > t_0$ such that

$$\gamma(t) < N, \text{ for } t \in R_+ \text{ and } \gamma(t) < \frac{\alpha}{2K}, \text{ for all } t \geq t^*.$$

From (13) we obtain

$$|\, \Psi(t) Z(t) \,| \leq K \,|\, \Psi(t_0) Z_0 \,|\, e^{-\alpha(t-t_0)+K\int_{t_0}^{t}\gamma(s)ds}, \text{ for } t \in [t_0, t_+). \tag{14}$$

Now, we have two cases further:
1). $t^* < t_+$
j). for $t \in [t_0, t^*]$, we have

$$-\alpha(t-t_0) + K\int_{t_0}^{t}\gamma(s)ds \leq -\alpha(t-t_0) + KN(t-t_0) \leq -\frac{\alpha}{2}(t-t_0) + KNt^*.$$

jj). for $t \in (t^*, t_+)$, we have

$$-\alpha(t-t_0) + K\int_{t_0}^{t}\gamma(s)ds = -\alpha(t-t_0) + K\int_{t_0}^{t^*}\gamma(s)ds + K\int_{t^*}^{t}\gamma(s)ds \leq$$

$$\leq -\alpha(t-t_0) + KN(t^*-t_0) + K\cdot\tfrac{\alpha}{2K}(t-t^*) \leq$$

$$\leq -\alpha(t-t_0) + KN(t^*-t_0) + \tfrac{\alpha}{2}(t-t_0) \leq -\tfrac{\alpha}{2}(t-t_0) + KNt^*.$$

2). $t^* \geq t_+$.
For $t \in [t_0, t_+)$, we have

$$-\alpha(t-t_0) + K\int_{t_0}^{t}\gamma(s)ds \leq -\alpha(t-t_0) + KN(t-t_0) < -\frac{\alpha}{2}(t-t_0) + KNt^*.$$

From (14) and the above results, we obtain

$$|\, \Psi(t) Z(t) \,| \leq Ke^{KNt^*} \,|\, \Psi(t_0) Z_0 \,|\, e^{-\frac{\alpha}{2}(t-t_0)}, \text{ for } t \in [t_0, t_+). \tag{15}$$

Thus, in all three cases i) - iii), the inequality obtained for $|\, \Psi(t) Z(t) \,|$ shows that $t_+ = +\infty$ and hence, the solution Z(t) is defined on R_+.
Now, we see from these results that the trivial solution of the equation (1) is $\Psi-$uniformly asymptotically stable on R_+. ■

Remark 4.1 *Theorem generalizes Theorem 9, [[3], Ch. III] from systems of differential equations to a nonlinear matrix differential equation.*

The theorem contains as a particular case a result concerning $\Psi-$uniform asymptotic stability of systems of differential equations of the form

$$z' = A(t)z + f(t,z) \tag{16}$$

as follows (See Corollary 3.1):

Theorem 4.2 *Suppose that:*
1). The hypothesis (H_1) adapted for systems of differential equations (16) is satisfied;
2). The equation $z' = A(t)z$ is $\Psi-$uniformly asymptotically stable on R_+;
3). The matrix function $f : R_+ \times R^n \to R^n$ satisfies the inequality

$$| \Psi(t)f(t,z) |\leq \gamma(t) | \Psi(t)z |$$

for all $t \in R_+$ and $z \in R^n$, where $\gamma : R_+ \to R_+$ is a continuous function that satisfies one of the following conditions:
i). $M = \sup\limits_{t\geq 0} \gamma(t)$ is a sufficiently small number;
ii). $L = \int_0^\infty \gamma(t)dt < +\infty$;
iii). $\lim\limits_{t\to\infty} \gamma(t) = 0$.
Then, the trivial solution of the nonlinear system of differential equations (16) is $\Psi-$uniformly asymptotically stable on R_+.

Remark 4.2 *This Theorem generalizes the Theorem 9, [[3], Ch. III] from uniform asymptotic stability of systems of differential equations to $\Psi-$uniform asymptotic stability of the systems of differential equations.*

Theorem 4.3 *Suppose that:*
1). The hypothesis (H_2) is satisfied;
2). The equation (4) is $\Psi-$uniformly asymptotically stable on R_+;
3). The matrix function $B : R_+ \to \mathbb{M}_{n\times n}$ is continuous on R_+;
4). The matrix function $F : R_+ \times \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ satisfies the inequality

$$| \Psi(t)F(t,Z) |\leq \gamma(t) | \Psi(t)Z |$$

for all $t \in R_+$ and $Z \in \mathbb{M}_{n\times n}$, where $\gamma : R_+ \to R_+$ is a continuous function that satisfies one of the following conditions:
i). $M = \sup\limits_{t\geq 0} (| B(t) | +\gamma(t))$ is a sufficiently small number;
ii). $L = \int_0^\infty (| B(t) | +\gamma(t))\, dt < +\infty$;
iii). $\lim\limits_{t\to\infty} (| B(t) | +\gamma(t)) = 0$.
Then, the trivial solution of the nonlinear matrix differential equation (2) is $\Psi-$uniformly asymptotically stable on R_+.

Proof. In Theorem (4.1) we consider $ZB(t) + F(t,Z)$ instead of $F(t,Z)$. ∎

Theorem 4.4 *Suppose that:*
1). The hypothesis (H_2) is satisfied;
2). The Lyapunov matrix differential equation (6) is $\Psi-$uniformly asymptotically stable on R_+;
3). The matrix function $F : R_+ \times \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ satisfies the inequality

$$| \Psi(t)F(t,Z) |\leq \gamma(t) | \Psi(t)Z |$$

for all $t \in R_+$ *and* $Z \in \mathbb{M}_{n\times n}$*, where* $\gamma : R_+ \to R_+$ *is a continuous function that satisfies one of the following conditions:*
i). $M = \sup_{t\geq 0} \gamma(t)$ *is a sufficiently small number;*
ii). $L = \int_0^\infty \gamma(t)dt < +\infty$;
iii). $\lim_{t\to\infty} \gamma(t) = 0$.
Then, the trivial solution of the nonlinear Lyapunov matrix differential equation (2) is $\Psi-$*uniformly asymptotically stable on* R_+.

Proof. From Lemma 2.6 one know that the trivial solution of the nonlinear Lyapunov matrix differential equation (2) is $\Psi-$uniformly asymptotically stable on R_+ if and only if the trivial solution of corresponding Kronecker product system associated with equation (2), i.e. the system

$$z' = \left(I \otimes A(t) + B^T(t) \otimes I\right) z + f(t,z) \tag{17}$$

is $I \otimes \Psi-$uniformly asymptotically stable on R_+, where $z = \mathcal{V}ec(Z)$ and $f(t,z) = \mathcal{V}ec\,(F(t,Z))$.
Using Theorem 4.2, we will show that the trivial solution of the differential system (17) is $I \otimes \Psi-$uniformly asymptotically stable on R_+.
From hypothesis (1) and Lemma 2.4 we have that the hypothesis (1) of Theorem 4.2 is satisfied.
We see from hypothesis (2) and Lemma 2.6 that the hypothesis (2) of Theorem 4.2 is satisfied.
From Lemma 2.5,

$$\| (I \otimes \Psi(t))\, f(t,z) \|_{R^{n^2}} = \| (I \otimes \Psi(t))\, \mathcal{V}ec\,(F(t,Z)) \|_{R^{n^2}} \leq$$

$$\leq | \Psi(t)F(t,Z) | \leq \gamma(t) \mid \Psi(t)Z \mid \leq n\gamma(t) \, \| (I \otimes \Psi(t))\, \mathcal{V}ec\,(Z) \|_{R^{n^2}} =$$

$$= n\gamma(t) \, \| (I \otimes \Psi(t))\, z \|_{R^{n^2}} \ .$$

Thus, the hypothesis (3) of Theorem 4.2 is satisfied.
Now, from Theorem 4.2, the trivial solution of the system (17) is $I \otimes \Psi-$uniformly asymptotically stable on R_+.
From Lemma 2.6, it results that the trivial solution of the nonlinear Lyapunov matrix differential equation (2) is $\Psi-$uniformly asymptotically stable on R_+. ■

The following Example illustrates the Theorem.

Example 4.1 *Consider the nonlinear Lyapunov matrix differential equation (2) with*

$$A(t) = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} \quad , \quad B(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

and

$$F(t,Z) = \begin{pmatrix} \frac{\arctan z_1}{1+t^4} & \frac{z_2}{1+t^2} \\ e^{-t}\sin z_3 & e^{-t^2} \mid z_4 \mid \end{pmatrix},$$

where $Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$.
The fundamental matrices for the equations (4) and (5) are

$$X(t) = \begin{pmatrix} e^{3t} & -\frac{1}{2}e^{-t} \\ 0 & e^{-t} \end{pmatrix} \quad , \quad Y(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 2e^{-2t} \end{pmatrix}$$

respectively.
Consider $\Psi(t) = \begin{pmatrix} e^{-4t} & 0 \\ 0 & 1 \end{pmatrix}$.
We have

$$\Psi(t)\, X(t)X^{-1}(s)\Psi^{-1}(s) = e^{-(t-s)} \begin{pmatrix} 1 & \frac{1}{2}\left(e^{-2s} - e^{-4t}\right) \\ 0 & 1 \end{pmatrix},\ t \geq s \geq 0$$

$$Y(t)\left(Y^T\right)^{-1}(s) = \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & e^{-2(t-s)} \end{pmatrix},\ t \geq s \geq 0$$

and hence

$$\Omega(t,s) = \left(Y(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)\, X(t)X^{-1}(s)\Psi^{-1}(s)\right) =$$

$$= e^{-\tau} \begin{pmatrix} e^{-\tau} & \lambda e^{-\tau} & 0 & 0 \\ 0 & e^{-\tau} & 0 & 0 \\ 0 & 0 & e^{-2\tau} & \lambda e^{-2\tau} \\ 0 & 0 & 0 & e^{-2\tau} \end{pmatrix}$$

where $\tau = t - s$ *and* $\lambda = \frac{e^{-2s} - e^{-4t}}{2}$.
It follows that $|\ \Omega(t,s)\ | \leq 2e^{-(t-s)}$, *for* $t \geq s \geq 0$. *From this and Theorem 3.2 it follows that the Lyapunov matrix differential equation (6) is* $\Psi-$*uniformly asymptotically stable on* R_+.
Further, the matrix function $F(t,Z)$ *satisfies the inequality*

$$|\ \Psi(t)F(t,Z)\ | \leq \gamma(t)\ |\ \Psi(t)Z\ |$$

for all $t \in R_+$ *and* $Z \in \mathbb{M}_{2\times 2}$, *where* $\gamma : R_+ \to R_+$ *is the continuous function*

$$\gamma(t) = \max\{\frac{1}{1+t^4} + \frac{1}{1+t^2}, e^{-t} + e^{-t^2}\}$$

that satisfies the condition $L = \int_0^{\infty} \gamma(t)dt < +\infty$.
From these, it is easy to see that the function F satisfies all the hypotheses of Theorem 4.4.
It is easy to see that the hypothesis (H_2) *is satisfied;*
Thus, the trivial solution of the nonlinear Lyapunov matrix differential equation (2) considered is $\Psi-$*uniformly asymptotically stable on* R_+.

Theorem 4.5 *Suppose that:*
1). The hypothesis (H_2) *is satisfied;*
2). The equation (5) is $\Psi-$*uniformly asymptotically stable on* R_+;
3). The matrix function $A : R_+ \to \mathbb{M}_{n\times n}$ *is continuous on* R_+;
4). The matrix function $F : R_+ \times \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ *satisfies the inequality*

$$|\ \Psi(t)F(t,Z)\ | \leq \gamma(t)\ |\ \Psi(t)Z\ |$$

for all $t \in R_+$ and $Z \in \mathbb{M}_{n\times n}$, where $\gamma : R_+ \to R_+$ is a continuous function that satisfies one of the following conditions:
i). $M = \sup\limits_{t\geq 0}\left(\gamma(t)+ \mid \Psi(t)A(t)\Psi^{-1}(t) \mid\right)$ is a sufficiently small number;
ii). $L = \int_0^{\infty}\left(\gamma(t)+ \mid \Psi(t)A(t)\Psi^{-1}(t) \mid\right)dt < +\infty$;
iii). $\lim\limits_{t\to\infty}\left(\gamma(t)+ \mid \Psi(t)A(t)\Psi^{-1}(t) \mid\right) = 0$.
Then, the trivial solution of the nonlinear Lyapunov matrix differential equation (2) is $\Psi-$uniformly asymptotically stable on R_+.

Proof. We see the nonlinear Lyapunov matrix differential equation (2) in the form

$$Z' = ZB(t) + A(t)Z + +F(t, Z)$$

and the proof goes through as for above Theorem. ■

Theorem 4.6 *Suppose that:*
1). The hypothesis (H_2) is satisfied;
2). The matrix differential equation $Z' = O$ is $\Psi-$uniformly asymptotically stable on R_+;
3). The matrix functions $A, B : R_+ \to \mathbb{M}_{n\times n}$ are continuous on R_+;
4). The matrix function $F : R_+ \times \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ satisfies the inequality

$$\mid \Psi(t)F(t, Z) \mid\leq \gamma(t) \mid \Psi(t)Z \mid$$

for all $t \in R_+$ and $Z \in \mathbb{M}_{n\times n}$, where $\gamma : R_+ \to R_+$ is a continuous function that satisfies one of the following conditions:
i). $M = \sup\limits_{t\geq 0}\left(\gamma(t)+ \mid \Psi(t)A(t)\Psi^{-1}(t) \mid + \mid B(t) \mid\right)$ is a sufficiently small number;
ii). $L = \int_0^{\infty}\left(\gamma(t)+ \mid \Psi(t)A(t)\Psi^{-1}(t) \mid + \mid B(t) \mid\right)dt < +\infty$;
iii). $\lim\limits_{t\to\infty}\left(\gamma(t)+ \mid \Psi(t)A(t)\Psi^{-1}(t) \mid + \mid B(t) \mid\right) = 0$.
Then, the trivial solution of the nonlinear Lyapunov matrix differential equation (2) is $\Psi-$uniformly asymptotically stable on R_+.

Proof. The proof is similar in spirit to that of above Theorems. ■

Theorem 4.7 *Suppose that:*
1). The hypothesis (H_3) is satisfied;
2). The equation (6) is $\Psi-$uniformly asymptotically stable on R_+;
3). The matrix function $G : R_+ \times R_+ \times \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ satisfies the inequality

$$\mid \Psi(t)G(t, s, Z) \mid\leq g(t - s) \mid \Psi(t)Z \mid$$

for all $0 \leq s \leq t < +\infty$ and $Z \in \mathbb{M}_{n\times n}$, where $g : R_+ \to R_+$ is a continuous function that satisfies the following conditions:
i). $\int_0^{\infty} g(t)dt$ is a sufficiently small number;
ii). $\lim\limits_{t\to\infty} g(t) = 0$.
Then, the trivial solution of the nonlinear Lyapunov matrix differential equation (3) is $\Psi-$uniformly asymptotically stable on R_+.

Proof. From Lemma 2.6 we know that the trivial solution of equation (3) is $\Psi-$uniformly asymptotically stable on R_+ if and only if the trivial solution of corresponding Kronecker product system associated with equation (3), i.e. the system

$$z' = \left(I \otimes A(t) + B^T(t) \otimes I\right)z + \int_0^t \gamma(t, s, z)ds, \tag{18}$$

where $z = \mathcal{V}ec(Z)$ and $\gamma(t,s,z) = \mathcal{V}ec\,(G(t,s,Z))$, is $I \otimes \Psi-$uniformly asymptotically stable on $\mathbb{R}_+$.
Using Lemma 2.8, we will show that the trivial solution of the differential system (18) is $I \otimes \Psi-$uniformly asymptotically stable on $\mathbb{R}_+$.
From hypothesis (1), Lemma 2.5 and Lemma 2.4, it follows that the hypothesis (1) of Lemma 2.8 is satisfied,with $I \otimes \Psi$ in role of Ψ, etc.
From hypothesis (2) and Theorem 3.2, for the fundamental matrix $U(t) = Y^T(t) \otimes X(t)$ of linear differential system associated with (18) (where X(t) and Y(t) are fundamental matrices for the equations (4) and (5) respectively), there exist $K > 0$ and $\alpha > 0$ such that

$$| (I \otimes \Psi(t)) \left(U(t)U^{-1}(s)\right) (I \otimes \Psi(s))^{-1} | \leq Ke^{-\alpha(t-s)}, \text{ for } 0 \leq s \leq t < +\infty.$$

Thus, the hypothesis (2) of Lemma 2.8 is satisfied, with $I \otimes \Psi$ in role of Ψ and $U(t)$ in role of X.
We will suppose that

$$\int_0^\infty | g(t) | \, dt < \frac{\alpha}{nK}.$$

From Lemma 2.5, we have that

$$\| (I \otimes \Psi(t))\, \gamma(t,s,z) \|_{R^{n^2}} = \| (I \otimes \Psi(t))\, \mathcal{V}ec\,(G(t,s,Z)) \|_{R^{n^2}} \leq$$

$$\leq | \Psi(t) G(t,s,Z) | \leq g(t-s) \, | \Psi(t) Z | \leq ng(t-s) \, \| (I \otimes \Psi(t))\, \mathcal{V}ec\,(Z) \|_{R^{n^2}} =$$

$$= ng(t-s) \, \| (I \otimes \Psi(t))\, z \|_{R^{n^2}} \, .$$

Thus, the hypothesis (3) of Lemma 2.8 is satisfied, with $I \otimes \Psi$ in role of Ψ, $\gamma(t,s,z)$ in role of $f(t,s,x)$ and ng in role of k, etc.
Thus, applying the Lemma 2.8, it follows that the trivial solution of the differential system (18) is $I \otimes \Psi-$uniformly asymptotically stable on $\mathbb{R}_+$.
Now, from Lemma 2.6, we have that the trivial solution of the nonlinear Lyapunov matrix differential equation (3) is $\Psi-$uniformly asymptotically stable on $\mathbb{R}_+$. ■

Remark 4.3 *Theorem 4.7 generalizes Theorem 5.1, [7] and Theorem 4, [5].*

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