ITM Web of Conferences **49**, 02005 (2022) https://doi.org/10.1051/itmconf/20224902005
Fourth ICAMNM 2022

Optimal control of compound Poisson processes

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Abstract. The problem of controlling a compound Poisson process until it leaves an interval is considered. This type of problem is known as a homing problem. To determine the value of the optimal control, we must solve a non-linear integro-differential equation. Exact and explicit solutions are obtained for two possible jumps size distributions.

1 Introduction

Let $\{X_u(t), t \geq 0\}$ be the controlled jump-diffusion process defined by

$$X_u(t) = X_u(0) + \mu t + \int_0^t b[X_u(s)]\, u[X_u(s)]\, \mathrm{d}s + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, $b(\cdot)$ is a non-zero function, $u(\cdot)$ is the control variable, $\{B(t), t \geq 0\}$ is a standard Brownian motion and $\{N(t), t \geq 0\}$ is a Poisson process (independent of $\{B(t), t \geq 0\}$) with rate λ. Moreover, the random variables $Y_1, Y_2, \ldots$ are independent and identically distributed.

In [1], the author considered the following problem: find the control that minimizes the expected value of the cost criterion

$$J(x) := \int_0^{T(x)} \left\{ \frac{1}{2}\, q[X_u(t)]\, u^2[X_u(t)] + \theta \right\} \mathrm{d}t + K[X_u(T(x))], \tag{2}$$

where θ is a real constant, $q(\cdot)$ is a positive function, K is a general terminal cost function and the final time $T(x)$ is a random variable (called a *first-passage time*) defined by

$$T(x) = \inf\{t \geq 0 : X_u(t) \notin (a, b) \mid X_u(0) = x \in [a, b]\}. \tag{3}$$

He was able to find explicit solutions to particular problems when the random variables $Y_1, Y_2, \ldots$ are exponentially distributed, so that the jumps are positive. This type of problem is known as a *homing problem* (see [3] and/or [4]). If the parameter θ is positive, then the optimizer must try to minimize the time that the controlled process spends in the *continuation region* (a, b), whereas the objective is to maximize the time that it spends in (a, b) when $\theta < 0$. In both cases, the optimizer must of course take the quadratic control costs $\frac{1}{2}\, q[X_u(t)]\, u^2[X_u(t)]$ and the termination cost $K[X_u(T(x))]$ into account.

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In the current paper, we set $\mu = \sigma = 0$. That is, $\{X_u(t), t \geq 0\}$ becomes a controlled *compound Poisson process*; see, for instance, [2]. Furthermore, as in the particular problems solved in [1], we assume that the ratio $b^2(x)/q(x)$ is constant:

$$\kappa := \frac{b^2(x)}{2\,q(x)} \quad (> 0). \tag{4}$$

Finally, we take $[a, b] = [0, 1]$ and we define

$$T(x) = \inf\{t \geq 0 : X_u(t) \geq 1 \mid X_u(0) = x \in [0, 1]\}. \tag{5}$$

Thus, there is now a single barrier, at $x = 1$.

To solve the above stochastic optimal control problem, we can use *dynamic programming*. First, we define the *value function*

$$F(x) = \inf_{\substack{u[X_u(t)] \\ 0 \leq t \leq T(x)}} E[J(x)]. \tag{6}$$

The function $F(x)$ is thus the expected cost (or reward, if it is negative) obtained by choosing the optimal value of the control $u[X_u(t)]$ for $t \in [0, T(x)]$. It satisfies a non-linear integro-differential equation that must be solved to determine the optimal control $u^*(x)$, which is expressed in terms of $F(x)$:

$$u^*(x) = -\frac{b(x)}{q(x)} F'(x). \tag{7}$$

In the next section, we will give the integro-differential equation satisfied by $F(x)$ and we will show that it can, in some cases, be transformed into a non-linear ordinary differential equation. We will then solve exactly and explicitly two particular problems. We will give a few concluding remarks in Section 3.

2 Optimal control

We deduce from Proposition 2.1 in [1] the following result.

Proposition 2.1 *The function $F(x)$ defined in Eq. (6) satisfies the first-order, non-linear integro-differential equation*

$$0 = \theta - \kappa\,[F'(x)]^2 - \lambda F(x) + \lambda \int_{-\infty}^{\infty} F(x + y)\, f_Y(y)\,\mathrm{d}y \tag{8}$$

for $0 \leq x < 1$, where $f_Y(y)$ is the common density function of the random variables $Y_1, Y_2, \ldots$ Moreover, we have the boundary condition

$$F(x) = K(x) \quad \textit{if } x \geq 1. \tag{9}$$

If Y has an exponential distribution with parameter α, so that the function $f_Y(y)$ is given by

$$f_Y(y) = \alpha\, e^{-\alpha y} \quad \text{for } y \geq 0, \tag{10}$$

then, proceeding as in [1], we can find particular solutions of (8), (9). In this paper, we will find new exact and explicit solutions. We assume that the jumps are x-dependent and bounded:

$$f_Y(y) = f_Y(y \mid x) = \frac{g(y)}{\int_{-x}^{1-x} g(w)\,\mathrm{d}w} \quad \text{for } -x \leq y \leq 1 - x, \tag{11}$$

where $g(y)$ is a positive function. It follows that

$$\int_{-x}^{1-x} f_Y(y \mid x)\,\mathrm{d}y = 1. \tag{12}$$

Hence, $f_Y(y \mid x)$ is a legitimate probability density function.

Remark 2.1 *Notice that if $u[X_u(s)] \equiv 0$, then the process $\{X_u(t), t \geq 0\}$ will remain inside the interval $[0,1]$ indefinitely. Therefore, we must assume that the parameter θ in the cost function $J(x)$ defined in Eq. (2) is positive; indeed, if $\theta < 0$, we will have $F(x) = -\infty$ by choosing $u[X_u(s)] \equiv 0$.*

We assume further that the function $g(y)$ is such that

$$g(y_1 + y_2) = g(y_1)\,g(y_2). \tag{13}$$

Then,

$$\begin{aligned}
\int_{-\infty}^{\infty} F(x+y)\,f_Y(y)\,\mathrm{d}y &= \int_{-x}^{1-x} F(x+y)\,f_Y(y \mid x)\,\mathrm{d}y \\
&\overset{z:=x+y}{=} \int_0^1 F(z)\,f_Y(z-x \mid x)\,\mathrm{d}z \\
&= \frac{g(-x)}{\int_{-x}^{1-x} g(w)\,\mathrm{d}w} \int_0^1 F(z)\,g(z)\,\mathrm{d}z. \qquad (14)
\end{aligned}$$

Notice that the integral involving the unknown function $F(z)$ in the above equation does not depend on x.

Case I. Suppose first that

$$g(y) = e^{-\alpha y}, \tag{15}$$

where $\alpha > 0$. This function is indeed such that Eq. (13) is satisfied. Moreover, we have

$$\frac{g(-x)}{\int_{-x}^{1-x} g(w)\,\mathrm{d}w} = \frac{\alpha\, e^{\alpha x}}{e^{\alpha x} - e^{\alpha(x-1)}} = \frac{\alpha}{1 - e^{-\alpha}}, \tag{16}$$

which is independent of x. Hence, the integro-differential equation (8) reduces to

$$0 = \theta - \kappa\,[F'(x)]^2 - \lambda F(x) + \frac{\lambda\alpha}{1-e^{-\alpha}} \int_0^1 F(z)\,e^{-\alpha z}\,\mathrm{d}z, \tag{17}$$

which can be rewritten as the Riccati equation

$$0 = \theta^* - \kappa\,[F'(x)]^2 - \lambda F(x), \tag{18}$$

where θ^* is a constant that actually depends on the unknown function $F(x)$.

Differentiating Eq. (18), we obtain

$$0 = -2\kappa F'(x)F''(x) - \lambda F'(x). \tag{19}$$

This non-linear ordinary differential equation has two solutions: $F(x) \equiv c$ and

$$F(x) = -\frac{\lambda}{4\kappa}x^2 + c_1 x + c_0, \tag{20}$$

where c, c_0 and c_1 are arbitrary constants. Because we assumed that the parameter θ is positive, the function $F(x) \equiv c$ does not satisfy Eq. (17). Substituting the function defined in Eq. (20) into (17), we find that this equation is satisfied if and only if

$$\begin{aligned} 0 \;=\; & 4c_1^2\kappa^2\alpha^2\left(1-e^{-\alpha}\right)+4c_1\kappa\lambda\alpha\left[(\alpha+1)e^{-\alpha}-1\right] \\ & -4\theta\kappa\alpha^2\left(1-e^{-\alpha}\right)+\lambda^2\left[2-(\alpha^2+2\alpha+2)e^{-\alpha}\right]. \end{aligned} \tag{21}$$

As an illustrative example, let us take $\alpha = \lambda = \kappa = \theta = 1$. Equation (21) reduces to

$$4c_1^2\left(1-e^{-1}\right)+4c_1\left(2e^{-1}-1\right)-2-e^{-1}=0. \tag{22}$$

The two roots of the above equation are $c_1 \simeq 1.199$ and $c_1 \simeq -0.781$. Therefore, we have two possible solutions:

$$F(x) \simeq -\frac{1}{4}x^2+1.199\,x+c_0 \quad \text{and} \quad F(x) \simeq -\frac{1}{4}x^2-0.781\,x+c_0. \tag{23}$$

Now, because $\theta > 0$, we can state that the function $F(x)$ should decrease when x increases in the interval $[0, 1)$. It follows that we must choose the solution

$$F(x) \simeq -\frac{1}{4}x^2-0.781\,x+c_0 \quad \text{for } 0 \le x < 1. \tag{24}$$

The constant c_0 is uniquely determined from the boundary condition $F(1) = K(1)$. Finally, the optimal control is given by

$$u^*(x) \simeq \frac{b(x)}{q(x)}\left(\frac{1}{2}x+0.781\right). \tag{25}$$

Remark 2.2 *Notice that the ratio $b^2(x)/q(x)$ was assumed to be equal to the constant 2κ. However, the ratio $b(x)/q(x)$ is not necessarily a constant. When it is indeed a constant, the optimal control is an affine function of x.*

Case II. Suppose next that

$$g(y) \equiv 1, \tag{26}$$

so that Eq. (13) is again satisfied. We calculate

$$\frac{g(-x)}{\int_{-x}^{1-x} g(w)\,\mathrm{d}w} \equiv 1. \tag{27}$$

The integro-differential equation (8) becomes

$$0 = \theta - \kappa\left[F'(x)\right]^2 - \lambda F(x) + \lambda\int_0^1 F(z)\,\mathrm{d}z. \tag{28}$$

This equation can be rewritten as Eq. (18). It follows that we can state that the function $F(x)$ is of the form given in Eq. (20).

The equation that corresponds to Eq. (21) is

$$0 = -c_1^2\kappa + c_1\frac{\lambda}{2} + \theta - \frac{\lambda^2}{12\kappa}. \tag{29}$$

With $\lambda = \kappa = \theta = 1$, the above equation is

$$0 = -c_1^2 + \frac{1}{2}c_1 + \frac{11}{12}. \tag{30}$$

We have the two roots $c_1 = \frac{1}{4} \pm \frac{\sqrt{141}}{12}$. As in Case I, we must choose the negative root, so that

$$F(x) = -\frac{1}{4}x^2 + \left(\frac{1}{4} - \frac{\sqrt{141}}{12}\right)x + c_0 \quad \text{for } 0 \leq x < 1. \tag{31}$$

The optimal control is given by

$$u^*(x) = \frac{b(x)}{q(x)}\left(\frac{1}{2}x - \frac{1}{4} + \frac{\sqrt{141}}{12}\right). \tag{32}$$

Remark 2.3 *Case II is the limiting case when α decreases to zero in Case I. Moreover, we can generalize the condition in Eq. (13) to*

$$g(y_1 + y_2) \propto g(y_1)g(y_2). \tag{33}$$

Then, in Case I, we could define

$$g(y) = \beta e^{-\alpha y}, \tag{34}$$

where $\beta > 0$, and

$$g(y) \equiv g_0 > 0 \tag{35}$$

in Case II (that is, Case I with $\alpha = 0$ and $\beta = g_0$).

3 Concluding remarks

We considered a special case of the stochastic optimal control problem studied by the author in [1] for jump-diffusion processes, but with a more general distribution for the jumps size. This special case is interesting in itself because in some applications the controlled process $\{X_u(t), t \geq 0\}$ has no continuous part. The (positive) random variables $Y_1, Y_2, \ldots$ could represent the claims received by an insurance company. The number of claims in a given interval is a discrete-state stochastic process.

We were able to obtain exact and explicit solutions to two particular problems, namely when the jumps size is distributed like an exponential random variable, but between $-x$ and $1-x$, where x is the value of the process at the moment of the jump, and also when the jumps size is uniformly distributed on the interval $[-x, 1-x]$.

When we are not able to obtain an exact solution to a given problem, we could at least try to find an approximate solution, perhaps in the form of a polynomial function. We could also use numerical methods to solve the integro-differential equation, subject to the appropriate boundary condition.

Acknowledgements

This research was supported by the Natural Sciences and Engineering Research Council of Canada.

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