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3D Quadratic ODE systems with an infinite number of limit cycles

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Abstract. We consider an autonomous three-dimensional quadratic ODE system with nine parameters, which is a generalization of the Langford system. We derive conditions under which this system has infinitely many limit cycles. First, we study the equilibrium points of such systems and their eigenvalues. Next, we prove the non-local existence of an infinite set of limit cycles emerging by means of Andronov – Hopf bifurcation.

1 Introduction

Three-dimensional autonomous systems of ODEs, in comparison with two-dimensional ones, demonstrate a much more complex qualitative behavior of solutions. In particular, three-dimensional autonomous systems of ODEs can have as attracting sets not only equilibrium points and limit cycles, but also strange or chaotic attractors [1–5]. It is known that even for quadratic systems, as the simplest case of planar systems, Hilbert's problem on the maximum number of limit cycles has not yet been solved, but the finiteness of such a number has been proved. As for three-dimensional systems, Bulgakov and Grin in [6] give an example of a such quadratic system with an infinite number (continuum) of limit cycles. In contrast to planar systems, where the limit cycle is considered as an isolated closed trajectory, to which all neighboring trajectories tend when $t \to +\infty$ or $t \to -\infty$, the behavior of trajectories in the vicinity of the limit cycle of a three-dimensional system can be more complex [7]. In particular, limit cycles may not be isolated in its phase space.

We consider a system

$$\begin{aligned} \dot{x} &= a_0 x - a_1 y + a_2 xy + a_3 y^2 + a_4 xz + a_5 yz, \\ \dot{y} &= a_1 x + a_0 y - a_2 x^2 - a_3 xy + a_4 yz - a_5 xz, \\ \dot{z} &= b_1 z - b_2 z^2 - b_3 \left(x^2 + y^2\right); \quad (x, y, z) \in \mathbb{R}^3, \end{aligned} \tag{1}$$

where $a_i, b_j \in \mathbb{R}$ ($i = \overline{0, 5}$, $j = \overline{1, 3}$) are system parameters. This system can be viewed as a generalization of the Langford system [5, p. 106], [8–10].

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Bulgakov and Grin [6] proved that a system of the form

$$\begin{aligned} \dot{x} &= a_0 x - a_1 y + a_2 xy + a_3 y^2 + a_4 xz + a_5 yz, \\ \dot{y} &= a_1 x + a_0 y - a_2 x^2 - a_3 xy + a_4 yz - a_5 xz, \\ \dot{z} &= 2\left(a_0 z + a_4 z^2\right); \quad (x, y, z) \in \mathbb{R}^3, \end{aligned} \tag{2}$$

where $a_i \in \mathbb{R}$ $(i = \overline{0,5})$ are system parameters, has infinitely many (continuum) limit cycles, which are born from the equilibrium point $O(0,0,0)$ and lie on the intersection of the family of surfaces $z = \left(x^2 + y^2\right)/k$, $k \in \mathbb{R}\backslash\{0\}$ and the plane $z = -a_0/a_4$. In this case, $x^2 + y^2 - kz = 0$ $(k \in \mathbb{R})$ and $a_4 z + a_0 = 0$ are the invariant surfaces for system (2). But the results presented in the mentioned paper [6] are local, since the facts used in the proof for the appearance of limit cycles due to the Andronov–Hopf bifurcation are valid only in a sufficiently small neighborhood of the focus $O(0,0,0)$ and for a sufficiently small change of the parameter a_0.

Bearing in mind the above, we are interested in what cases system (1) has similar invariant surfaces and infinitely many limit cycles which can exist in the phase space not only in the vicinity of the focus and not only for parameter values close to the bifurcation value.

2 Preliminary results

As the first step we find cases of the system (1) which have invariant surfaces in the form of elliptic paraboloids and a plane.

Theorem 2.1 *In order for system* (1) *to have the invariant surfaces* $f_1 = x^2 + y^2 - kz = 0$ $\forall k \in \mathbb{R}$ *and* $f_2 = z + r = 0$ *(*$r \in \mathbb{R}$*), it is necessary and sufficient that* $b_3 = 0$*,* $b_1 = 2a_0$*,* $b_2 = -2a_4$*,* $r(a_4 r - a_0) = 0$ *(in this case, system* (1) *takes the form* (2)*).*

Proof. Necessity. Let $\forall k \in \mathbb{R}$ $f_1 = x^2 + y^2 - kz = 0$ be the invariant surface for system (1). Therefore, the derivative of the function f_1 along trajectories of system (1) must be identically zero. The derivative of the function f_1 along trajectories of system (1) is $\dot{f}_1 = \frac{\partial f_1}{\partial x}\dot{x} + \frac{\partial f_1}{\partial y}\dot{y} + \frac{\partial f_1}{\partial z}\dot{z} = \left(x^2 + y^2\right)(2a_0 + kb_3 + 2a_4 z) - kzb_1 + kz^2 b_2$. Since $x^2 + y^2 = kz$, then $\dot{f}_1 = k\left((b_2 + 2a_4)z + 2a_0 - b_1 + kb_3\right)z \equiv 0$.

Let $f_2 = z + r = 0$ be the invariant surface for system (1). Therefore, the derivative of the function f_2 along trajectories of system (1) must be identically zero. The derivative of the function f_2 along trajectories of system (1) is $\dot{f}_2 = z(b_1 - zb_2) - \left(x^2 + y^2\right)b_3$. Since $z = -r$, then $\dot{f}_2 = -b_3\left(x^2 + y^2\right) - r(b_1 + rb_2) \equiv 0$. So, we have a system of equations

$$\begin{cases} k\left((b_2 + 2a_4)z + 2a_0 - b_1 + kb_3\right)z \equiv 0, \\ -b_3\left(x^2 + y^2\right) - r(b_1 + rb_2) \equiv 0, \end{cases}$$

from which (taking into account that the first identity must be satisfied $\forall k \in \mathbb{R}$) we obtain $b_3 = 0$, $b_1 = 2a_0$, $b_2 = -2a_4$ and $r(b_1 + rb_2) = 0$. The necessity has been proven.

Sufficiency. Suppose that conditions $b_3 = 0$, $b_1 = 2a_0$, $b_2 = -2a_4$, $r(a_4 r - a_0) = 0$ are valid i.e. system (1) has the form (2). The derivative of the function f_1 along trajectories of system (1) is $\dot{f}_1 = 2\left(x^2 + y^2 - kz\right)(a_0 + za_4)$. Whence it follows that $f_1 = x^2 + y^2 - kz = 0$ is the invariant surface for system (1). The derivative of the function f_2 along trajectories of system (1) is $\dot{f}_2 = 2z(a_0 + za_4)$. Whence (taking into account that $r(a_4 r - a_0) = 0$) it follows that $f_2 = z + r = 0$ is the invariant surface for system (1). □

2.1 Equilibrium points

Let us find the equilibrium points of system (2). Let's introduce the designations of points $P_1(0,0,0)$, $P_2\left(0,0,-\frac{a_0}{a_4}\right)$; straight lines $P_3\left(x, \frac{a_1a_4+a_0a_5}{a_3a_4} - \frac{a_2}{a_3}x, -\frac{a_0}{a_4}\right)$, where $x \in \mathbb{R}$, $P_4\left(\frac{a_1a_4+a_0a_5}{a_2a_4} - \frac{a_3}{a_2}y, y, -\frac{a_0}{a_4}\right)$, where $y \in \mathbb{R}$; and plane $P_5\left(x, y, -\frac{a_0}{a_4}\right)$, where $x, y \in \mathbb{R}$. Note that for $a_0 = 0$ the point P_2 merges with P_1.

Proposition 2.1 *For systems* (2)*, the following cases are possible.*

1) If $a_4 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$, then we have two equilibrium points P_1, P_2 and an equilibrium line P_3 (or P_4). Note that for $a_1 = -\frac{a_0a_5}{a_4}$ the point $P_2 \in P_3$ (or P_4).

2) If $a_4 \neq 0$, $a_2 \neq 0$, $a_3 = 0$, $a_1 \neq -\frac{a_0a_5}{a_4}$, then we have two equilibrium points P_1, P_2 and an equilibrium line P_4.

3) If $a_4 \neq 0$, $a_2 \neq 0$, $a_3 = 0$, $a_1 = -\frac{a_0a_5}{a_4}$, then we have an equilibrium point P_1 and an equilibrium line P_4. Note that for $a_0 = 0$ the point $P_1 \in P_4$.

4) If $a_4 \neq 0$, $a_3 \neq 0$, $a_2 = 0$, $a_1 \neq -\frac{a_0a_5}{a_4}$, then we have two equilibrium points P_1, P_2 and an equilibrium line P_3.

5) If $a_4 \neq 0$, $a_3 \neq 0$, $a_2 = 0$, $a_1 = -\frac{a_0a_5}{a_4}$, then we have an equilibrium point P_1 and an equilibrium line P_3. Note that for $a_0 = 0$ the point $P_1 \in P_3$.

6) If $a_4 \neq 0$, $a_2 = a_3 = 0$, $a_1 \neq -\frac{a_0a_5}{a_4}$, then we have two equilibrium points P_1 and P_2.

7) If $a_4 \neq 0$, $a_2 = a_3 = 0$, $a_1 = -\frac{a_0a_5}{a_4}$, then we have an equilibrium point P_1 and an equilibrium plane P_5. Note that for $a_0 = 0$ the point $P_1 \in P_5$.

8) If $a_4 = 0$, $a_0 \neq 0$ then we have an equilibrium point P_1.

Notice, that for $a_0 \neq 0$ only under the condition $a_1 \neq -\frac{a_0a_5}{a_4}$ there is a neighborhood of the point P_2 such that at its intersection with the plane $z = -a_0/a_4$ there are no other equilibrium points of system (2) . This is realized in case 1), under the additional condition $a_1 \neq -\frac{a_0a_5}{a_4}$, and in cases 2), 4), 6) of Proposition 2.1. Moreover, in case 6), the above-mentioned neighborhood can be considered the entire space $\mathbb{R}^3$.

For system (2), the Jacobi matrix has the form

$$\begin{pmatrix} a_0 + a_2y + a_4z & -a_1 + a_2x + 2a_3y + a_5z & a_4x + a_5y \\ a_1 - 2a_2x - a_3y - a_5z & a_0 - a_3x + a_4z & a_4y - a_5x \\ 0 & 0 & 2(a_0 + 2a_4z) \end{pmatrix}.$$

Let us find the eigenvalues for the equilibrium points of system (2).

Proposition 2.2 *1) For the equilibrium point P_1 of system* (2)*, the eigenvalues are $\lambda_1 = 2a_0$, $\lambda_{2,3} = a_0 \pm a_1 i$. If $a_0 < 0$, then P_1 is a stable focus, if $a_0 > 0$, then P_1 is an unstable focus.*

2) For the equilibrium point P_2 of system (2)*, the eigenvalues are $\lambda_1 = -2a_0$, $\lambda_{2,3} = \pm\left(a_1 + \frac{a_0a_5}{a_4}\right)i$.*

3) For the equilibrium line P_3 of system (2)*, the eigenvalues are $\lambda_1 = -2a_0$, $\lambda_2 = 0$, $\lambda_3 = \frac{a_1a_4+a_0a_5}{a_3a_4}a_2 - \frac{a_2^2+a_3^2}{a_3}x$. If $x = \frac{a_2(a_1a_4+a_0a_5)}{a_4(a_2^2+a_3^2)}$, then $\lambda_3 = 0$. For $x < \frac{a_2(a_1a_4+a_0a_5)}{a_4(a_2^2+a_3^2)}$, the eigenvalue $\begin{cases} \lambda_3 > 0, & if \quad a_3 > 0, \\ \lambda_3 < 0, & if \quad a_3 < 0. \end{cases}$ If $x > \frac{a_2(a_1a_4+a_0a_5)}{a_4(a_2^2+a_3^2)}$, then $\begin{cases} \lambda_3 > 0, & for \quad a_3 < 0, \\ \lambda_3 < 0, & for \quad a_3 > 0. \end{cases}$*

4) *For the equilibrium line P_4 of system* (2), *the eigenvalues are* $\lambda_1 = -2a_0$, $\lambda_2 = 0$, $\lambda_3 = \frac{a_2^2+a_3^2}{a_2}y - \frac{a_1a_4+a_0a_5}{a_2a_4}a_3$. *If* $y = \frac{a_3(a_1a_4+a_0a_5)}{a_4(a_2^2+a_3^2)}$, *then* $\lambda_3 = 0$. *If* $y < \frac{a_3(a_1a_4+a_0a_5)}{a_4(a_2^2+a_3^2)}$, *then* $\begin{cases} \lambda_3 > 0, & for \quad a_2 < 0, \\ \lambda_3 < 0, & for \quad a_2 > 0. \end{cases}$ *If* $y > \frac{a_3(a_1a_4+a_0a_5)}{a_4(a_2^2+a_3^2)}$, *then* $\begin{cases} \lambda_3 > 0, & for \quad a_2 > 0, \\ \lambda_3 < 0, & for \quad a_2 < 0. \end{cases}$

5) *For the equilibrium plane P_5 of system* (2) *(in the case $a_2 = a_3 = 0$, $a_1 = -\frac{a_0a_5}{a_4}$), the eigenvalues are $\lambda_1 = -2a_0$, $\lambda_{2,3} = 0$.*

3 Main results

Now we prove the nonlocal existence of an infinite number of limit cycles for system (2).

Theorem 3.1 *In the case $a_4 \neq 0$, $a_1 \neq -a_0a_5/a_4$ under the condition $a_0k/a_4 < 0$ infinitely many (continuum) limit cycles of the system* (2) *bifurcate from the focus $O(0,0,0)$ due to the Andronov – Hopf bifurcation, which are stable (unstable) if $a_0 > 0$, $\frac{a_4}{a_1k} < 0$ ($a_0 < 0$, $\frac{a_4}{a_1k} > 0$). Each of the limit cycles retains its existence with increasing $|a_0|$, as long as the condition $-\frac{a_0k}{a_4} < \frac{(a_1a_4+a_0a_5)^2}{a_4^2(a_2^2+a_3^2)}$ holds. If $a_2 = a_3 = 0$, then limit cycles exist for all values $|a_0| \neq 0$.*

Proof. The existence (according to Theorem 2.1) of the invariant surfaces

$$f_1 = x^2 + y^2 - kz = 0 \quad \forall k \in \mathbb{R} \tag{3}$$

for a third-order system (2) allows to reduce its study to the following second-order system

$$\begin{aligned} \dot{x} &= a_0x - a_1y + a_2xy + a_3y^2 + a_4x(x^2+y^2)/k + a_5y(x^2+y^2)/k, \\ \dot{y} &= a_1x + a_0y - a_2x^2 - a_3xy + a_4y(x^2+y^2)/k - a_5x(x^2+y^2)/k. \end{aligned} \tag{4}$$

Therefore, the proof of the existence of a limit cycle for the system (2) on an arbitrary surface of family (3) can be reduced to the corresponding second-order system (4). It has the focus at the point $O(0,0)$ with eigenvalues $\gamma_{2,3} = a_0 \pm a_1 i$ and the first focal Lyapunov quantity [7] $L_1 = \frac{2\pi a_4}{a_1k}$. For a sufficiently small change of the parameter a_0 and under the condition $a_0L_1 < 0$ in the vicinity of the focus $O(0,0)$ of system (4), a stable (unstable) limit cycle bifurcates if $a_0 > 0$, $L_1 < 0$ ($a_0 < 0$, $L_1 > 0$). Therefore, system (2) in the vicinity of the focus $O(0,0,0)$ on each surface $x^2 + y^2 - kz = 0$ has one such limit cycle too. This implies that for $\frac{2\pi a_4a_0}{a_1k} < 0$ system (2) has infinitely many (continuum) limit cycles bifurcating from the focus $O(0,0,0)$ due to the Andronov – Hopf bifurcation. Since the plane $a_0 = -a_4z$ is also an invariant surface of the system (2) (according to Theorem 2.1), then each limit cycle is the circle of intersection of this plane with one of the paraboloids (3). Geometrically, this means that all cycles of system (2) form a family of concentric circles and completely fill the plane $a_0 = -a_4z$ (see Figure 1). Thus, each of the considered limit cycles represents an isolated circle in the phase space of system (4), but in the phase space of system (2) this circle is not isolated.

Now, to show the nonlocal existence of the appeared limit cycles of system (2), we will prove the existence and uniqueness of the limit cycle for system (4) using the topographic system of functions $V = x^2 + y^2 = C$. The derivative of these functions along of the system (4)

$$\frac{dV}{dt} = 2\left(x^2+y^2\right)\left(a_0 + a_4(x^2+y^2)/k\right)$$

does not change sign on the entire phase plane if $a_0k/a_4 > 0$. Therefore, in this case system (4) does not have a limit cycle and, consequently, system (2) does not have a cycle on any of

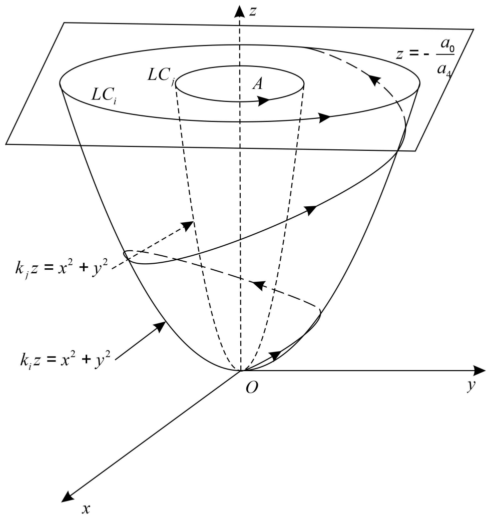

Figure 1. Localization of the limit cycles LC_i and LC_j as curves of intersection of the invariant plane with the family of invariant paraboloids in the phase space of system (2)

the paraboloids $x^2 + y^2 = kz$. In the case $a_0k/a_4 < 0$ the derivative changes its sign only on the circle

$$x^2 + y^2 = -a_0k/a_4, \tag{5}$$

which represents the unique limit cycle of the system (4). Accordingly, the same circle represents also the unique limit cycle of system (2) located on each paraboloid $x^2 + y^2 = kz$. Now we study the stability of this limit cycle using the sign of the derivative $\frac{dV}{dt}$. For $a_0 < 0$, $k/a_4 > 0$ the derivative $\frac{dV}{dt} < 0$ inside the circle (5) and $\frac{dV}{dt} > 0$ outside it, and consequently, the trajectories of the system (4) at $t \to +\infty$ enter the inside of those circles $V = C$ whose radii are less than the radius of the circle (5), and the trajectories go out of the circles $V = C$, whose radii are greater than the radius of the circle (5). This means that for $a_0 < 0$, $k/a_4 > 0$ the limit cycle is unstable. In the same way it is proved that for $a_0 > 0$, $k/a_4 < 0$ the limit cycle is stable. It is easy to see that the equilibrium point P_2 has the same character of stability as the limit cycles, and the equilibrium point P_1 has the opposite character of stability.

The proof of the nonlocal existence of the limit cycle and information about its stability for system (4) can also be obtained by writing it in polar coordinates

$$\begin{aligned}\frac{dr}{dt} &= r(a_0 + a_4r^2/k),\\ \frac{d\varphi}{dt} &= a_1 - a_2r\cos\varphi - a_3r\sin\varphi - a_5r^2/k.\end{aligned}$$

Next, we should take into account the influence of all equilibrium points of the system (2) derived before. For increasing $|a_0|$, the amplitude of limit cycles increases too, and they can disappear at some moment due to the influence of equilibrium points lying on the straight line P_3 or P_4 (in other words, on the line P : $a_2x + a_3y - (a_1a_4 + a_0a_5)/a_4 = 0$ in the plane $a_0 = -a_4z$). For such a destruction, the amplitude of the limit cycle must reach the straight line P which occurs when the following relation holds $-\frac{a_0k}{a_4} = \frac{(a_1a_4+a_0a_5)^2}{a_4^2(a_2^2+a_3^2)}$. Therefore, limit cycles exist as long as the condition $-\frac{a_0k}{a_4} < \frac{(a_1a_4+a_0a_5)^2}{a_4^2(a_2^2+a_3^2)}$ holds.

In particular, from the above study of the equilibrium points, it is obvious that this condition is true for the case $a_2 = a_3 = 0$ when the straight line P goes to infinity, and system (2) has only two equilibrium points P_1, P_2. Consequently, in this case all limit cycles for $|a_0| \to \infty$ expanding unboundedly, go to infinity. Thus, we have proved the assertion of the Theorem. □

4 Conclusion

For a three-dimensional autonomous ODE system (1), which is a generalization of the Langford system, we have obtained necessary and sufficient conditions under which system (1) has such invariant surfaces as elliptic paraboloids and a plane. Thus, we have come to system (2), which has an infinite number of cycles, which are the intersection of invariant elliptic paraboloids and the invariant plane. For system (2) we found the equilibrium points and calculated the eigenvalues for each of the equilibrium points. For system (2), under certain conditions on the parameters, it is proved that an infinite number (continuum) of limit cycles bifurcate from the focus $O(0, 0, 0)$ due to the Andronov – Hopf bifurcation. The nature of the stability of these limit cycles is also established.

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References

[1] M.W. Hirsch, S. Smale, R.L. Devaney, eds., *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, third edition edn. (Academic Press, Boston, 2013), ISBN 978-0-12-382010-5, `https://www.sciencedirect.com/science/article/pii/B9780123820105000257`

[2] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Springer New York, New York, NY, 2004), ISBN 978-1-4757-3978-7, `https://doi.org/10.1007/978-1-4757-3978-7_1`

[3] V.I. Arnold, V.S. Afraĭmovich, Y.S. Il'yashenko, L.P. Shil'nikov, *Bifurcation theory and catastrophe theory. Translated from the 1986 Russian original by N. D. Kazarinoff*, reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences (Dynamical systems V), 5 (1994) edn. (Berlin: Springer, 1999), ISBN 3-540-65379-1

[4] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos* (Springer New York, New York, NY, 2003), ISBN 978-0-387-21749-9, `https://doi.org/10.1007/0-387-21749-5_1`

[5] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, *Theory and applications of Hopf bifurcation*, Vol. 41 of *Lond. Math. Soc. Lect. Note Ser.* (Cambridge University Press, Cambridge. London Mathematical Society, London, 1981)

[6] V.I. Bulgakov, A.A. Grin', Differ. Equ. **32**, 1697 (1996)

[7] N. Bautin, E. Leontovich, *Methods and means for a qualitative investigation of dynamical systems on the plane* (Nauka, Moscow, 1990), (in Russian)

[8] E.V. Musafirov, Probl. Phys. Math. Tech. **28**, 47 (2016), (in Russian)

[9] E.V. Musafirov, Int. J. Bifurcation and Chaos **27**, 1750154 (2017)

[10] E. Musafirov, A. Grin, A. Pranevich, Int. J. Bifurcation and Chaos **32**, 2250038 (2022), `https://doi.org/10.1142/S0218127422500389`