Extended curvatures and Lie pseudoalgebras

Marcela Popescu¹,* and Paul Popescu¹,²,**

¹Department of Applied Mathematics, University of Craiova, DJ 200585, ROMANIA
²Doctoral School of Faculty of Applied Sciences, Politehnica University of Bucharest, 060042 Bucharest, Romania

Abstract. The aim of the paper is to prove that if some extended curvatures on a preinfinitesimal module $L$, considered in the paper, vanish, then the derived preinfinitesimal module $L^{(1)}$ is a Lie pseudoalgebra. Two non-trivial examples are given. The first example is when $L_0$ is an infinitesimal module and the second one is when $L_1$ is a preinfinitesimal module.

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1 Introduction

The Lie pseudoalgebras (see [5, 14, 22]) are generalizations of Lie algebroids [7, 13, 15, 25]. They are known also as Lie-Rinehart algebras (see, for example, [12]). Some less conditions in the definition of a Lie pseudoalgebra gives more general structures, known under different names and using different forms, but we use the most definitions in [21, 22, 25]. A general definition gives rise to anchored modules, that correspond to anchored vector bundles considered in [19], as relative tangent spaces; the infinitesimal modules correspond, via vector bundles setting, to skew symmetric Lie algebroids [17]; the preinfinitesimal modules correspond on vector bundles to almost Lie vector bundles, called as almost Lie structures in [16, 20, 23].

In [22] we define the (first) derived preinfinitesimal module of a given preinfinitesimal module, and we use it in the present paper. We suppose that a linear $L$-connexion $\nabla$ with a null torsion exists, using it in the background. Two extended curvatures of $\nabla$ are defined, and we prove that a sufficient condition that the derived preinfinitesimal module $L^{(1)}$ be a Lie pseudoalgebra is that both these two extended curvatures vanish (the main Theorem 3.1).

The main Theorem 3.1 is applied for two examples. In the first example $E_0$ is an infinitesimal module and the second one is when $E_1$ is not an infinitesimal module. The results are algebraic versions of vector bundle constructions, using Lie algebroids and skew-symmetric vector bundles [18].

2 Preinfinitesimal modules

Consider a commutative ring $k$ and and a commutative and associative $k$-algebra $A$. When $L$ is an $A$-module, we say also that $(A, L)$ is a module. For example, we have the module of

*e-mail: marcelacpopescu@yahoo.com
**e-mail: paul_p_popescu@yahoo.com

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derivations on $A$, $(A, \mathcal{D}er(A))$, where $\mathcal{D}er(A)$ is the $k$–module of maps $X : A \to A$, such that $X(a \cdot b) = X(a) \cdot b + a \cdot X(b)$.

A couple $(L, \rho)$ is an anchored module if $(A, L)$ is a module and $\rho : L \to \mathcal{D}er(A)$ an $A$-linear map, called an anchor (or an arrow). A triple $(L, \rho, [\cdot, \cdot]_L)$ is a preinfinitesimal module if $(L, \rho)$ is an anchored module and $[\cdot, \cdot]_L$ is a bracket, i.e. a $k$-bilinear map $[\cdot, \cdot]_L : L \times L \to L$ which is skew symmetric, i.e.

$$[Y, X]_L = -[X, Y]_L,$$

$(\forall)X, Y \in L$ and fulfills the property

$$[X, a \cdot Y]_L = \rho(X)(a) \cdot Y + a \cdot [X, Y]_L, (\forall)X, Y \in L, a \in A.$$

Consider now the maps

$$\mathcal{D} : L \times L \to \mathcal{D}er(A), \mathcal{D}(X, Y) = [\rho(X) \cdot \rho(Y)] - \rho([X, Y]_L)$$

and

$$\mathcal{J}_L : L^3 \to L, \mathcal{J}_L(X, Y, Z) = \sum_{\text{circ}} [X, [Y, Z]_L]_L.$$

As for example in [6]) $\mathcal{J}_L$ is called the Jacobiator of the bracket $[\cdot, \cdot]_L$.

In general, on a preinfinitesimal module $L$, $\mathcal{D}$ is $A$–bilinear and skew symmetric.

Proposition 2.1 Let $L$ be a preinfinitesimal module, with a skew-symmetric bracket. Then the following statements hold true:

1. The map $\mathcal{D}$ is $A$–bilinear and skew symmetric.
2. The map $\mathcal{J}_L$ is skew symmetric and $\mathcal{J}_L(X, Y, fZ) = \mathcal{D}(X, Y)(f)Z + f\mathcal{J}_L(X, Y, Z)$, $(\forall) f \in A$ and $X, Y, Z \in L$.
3. The map $\mathcal{J}_L$ is $A$–linear in all its arguments iff $L$ is an infinitesimal module.
4. The map $\mathcal{J}_L$ vanishes iff $L$ is a Lie pseudoalgebra.

In the particular case of infinitesimal modules we obtain as follows.

Proposition 2.2 For a preinfinitesimal module $L$ with a skew-symmetric bracket, the following conditions are equivalent:

1. $L$ is an infinitesimal module;
2. $\mathcal{D} = 0$;
3. $\mathcal{J}_L$ is $A$–linear in all its three arguments.

The Jacobiator $\mathcal{J}_L$ vanishes iff $L$ is a Lie pseudoalgebra.
The map \( \mathcal{D} \) can be seen as well as an anchor \( \rho^\wedge : L \wedge L \to \text{Der}(A) \),
\[
\rho^\wedge (X \wedge Y) = \mathcal{D}(X, Y),
\]
where \( L \wedge L \) denotes the exterior two product module of \( L \) over \( A \).

Consider now an anchored module \( L \) and let \((A, M)\) be a module. A linear \( L \)-connection on \( M \) is a map \( \nabla : L \times M \to M \) that verifies Koszul conditions:
\[
\begin{align*}
\nabla_{fX}s &= f \nabla_Xs, \\
\nabla_{X+X'}s &= \nabla_Xs + \nabla_{X'}s, \\
\nabla_X(fs) &= \rho(X)(f)s + f\nabla_Xs, \\
\nabla_X(s+s') &= \nabla_Xs + \nabla_Xs',
\end{align*}
\]
\((\forall)X, X' \in L, s, s' \in M, f \in A.\)

We suppose that such a linear \( L \)-connection \( \nabla \) exists.

Consider now that \( L \) is a preinfinitesimal module. The curvature of \( \nabla \) is the map \( R : L^2 \times M \to M \), given by the formula
\[
R(X, Y)s = \nabla_X\nabla_Ys - \nabla_Y\nabla_Xs - \nabla_{[X,Y]_L}s,
\]
\((\forall)X, Y \in L \) and \( s \in M.\)

**Proposition 2.3** Let \( L \) be a preinfinitesimal module, \((A, M)\) be a module and \( \nabla \) be a linear \( L \)-connection on \( M \). Then the formula \( R_{X,Y}s = R(X, Y)s \) defines a linear \( L \wedge L \)-connection on \( M \), according to the anchor \( \rho^\wedge \).

Now let us consider the particular case of an \( L \)-connection \( \nabla \) on an anchored module \( L \). In this case \([\cdot, \cdot]_L : L \times L \to L, [X, Y]_L \overset{\text{def}}{=} \nabla_XY - \nabla_YX,\) is a bracket on \( L \), so \( L \) is a preinfinitesimal module.

When \( L \) is a preinfinitesimal module, we can consider the torsion of an \( L \)-connection \( \nabla \) as a map
\[
T : L^2 \to L,
\]
given by
\[
T(X, Y) = \nabla_XY - \nabla_YX - [X, Y]_L.
\]

The curvature of \( \nabla \) is \( R : L^3 \to L \), given by
\[
R(X, Y, Z) = \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X,Y]_L}Z \overset{\text{not}}{=} \nabla_{X\wedge Y}Z,
\]
\((\forall)X, Y, Z \in L.\) Then \( R \) defines a linear \( L \wedge L \)-connection \( \nabla^\wedge \) on \( L \), according to the anchor \( \rho^\wedge \):
\[
\nabla^\wedge_{X\wedge Y}Z \overset{\text{def.}}{=} \nabla_{X\wedge Y}Z.
\]

Using the formula
\[
\nabla^\wedge_{X\wedge Y}(Z \wedge W) = \nabla_{X\wedge Y}Z \wedge W + Z \wedge \nabla_{X\wedge Y}W \overset{\text{not}}{=} \nabla_{X\wedge Y}(Z \wedge W)
\]
we get a linear \( L \wedge L \)-connection \( \nabla^{(\wedge)} \) on \( L \wedge L \), according to the anchor \( \rho^\wedge \).

Notice that when the torsion of \( \nabla \) vanishes, then
\[
[X, Y]_L = \nabla_XY - \nabla_YX.
\]
(1)
3 The derived preinfinitesimal module

Let \((\rho, L)\) be an anchored module such that there is an \(L\)-connexion \(\nabla\) on \(L\). Thus \(L\) is a preinfinitesimal module and \(\nabla\) has a null torsion.

On the module \(L^{(1)} = L \oplus (L \wedge L)\) there is an anchor given by the formula
\[
\rho^{(1)}(X + (Y \wedge Z)) = \rho(X) + \rho^\wedge(Y \wedge Z)
\]
and an \(L^{(1)}\)-connexion \(\nabla^{(1)}\) on \(L^{(1)}\), according to the formulas
\[
\nabla^{(1)}_XY = \nabla_XY + \frac{1}{2} X \wedge Y, \quad \nabla^{(1)}_X(Y \wedge Z) = \nabla_X(Y \wedge Z) = \nabla_XY \wedge Z + Y \wedge \nabla_XZ, \quad (2)
\]
\[
\nabla^{(1)}_{X\wedge Y}Z = \nabla_{X\wedge Y}Z, \quad \nabla^{(1)}_{X\wedge Y}(Z \wedge T) = \nabla_{X\wedge Y}(Z \wedge T).
\]

A bracket on \(L^{(1)}\) according to the anchor \(\rho^{(1)}\) can be given by
\[
[U, V]_{L^{(1)}} = \nabla^{(1)}_U V - \nabla^{(1)}_V U.
\]

More exactly, the bracket on \(L^{(1)}\) is given explicitly by the formulas
\[
[X, Y]_{L^{(1)}} = [X, Y]_L + X \wedge Y,
\]
\[
[X \wedge Y, Z]_{L^{(1)}} = \nabla_{X\wedge Y}Z - \nabla_Z(X \wedge Y),
\]
\[
[X \wedge Y, Z \wedge T]_{L^{(1)}} = \nabla_{X\wedge Y}(Z \wedge T) - \nabla_{Z\wedge T}(X \wedge Y). \quad (3)
\]

According to [22], the preinfinitesimal module \((L^{(1)}, \rho^{(1)}, [\cdot, \cdot]_{L^{(1)}})\) is called the (first) derived preinfinitesimal module of \((L, \rho)\), given by \(\nabla\).

**Proposition 3.1** Let us ‘lose the linear \(L\)-connexion \(\nabla\) on the preinfinitesimal module \(L\) has no torsion. Then the following properties hold true:

1. \(\mathcal{D}^{(1)}(X, Y) = [\rho^{(1)}(X), \rho^{(1)}(Y)] - \rho^{(1)}([X, Y]_{\theta^{(1)}}) = 0, \forall X, Y \in L\).
2. \(\mathcal{J}^{(1)}(X, Y, Z) = 0, \forall X, Y, Z \in L\), where \(\mathcal{J}^{(1)}\) is the Jacobiator of \([\cdot, \cdot]_{L^{(1)}}\).

We are going to show that the identical non-vanishing components of the Jacobiator on \(L^{(1)}\) can be expressed using two other curvatures involving \(\nabla\), that we define below.

For this, let us consider
\[
(X, Y, Z, W) \rightarrow R(X, Z \wedge W) Y \overset{\text{def}}{=} \nabla_X \nabla_{Z \wedge W} Y - \nabla_{Z \wedge W} \nabla_X Y = \nabla_{\nabla_X(Z \wedge W)} Y + \nabla_{\nabla_{Z \wedge W} X} Y,
\]
\[
R(Z \wedge W, X) Y \overset{\text{def}}{=} -R(X, Z \wedge W) Y
\]
and
\[
(X, Y, Z, W, U) \rightarrow R(X \wedge Y, Z \wedge W) U \overset{\text{def}}{=} \nabla_X \nabla_{Z \wedge W} U - \nabla_{Z \wedge W} \nabla_{X \wedge Y} U - \nabla_{\nabla_X(Z \wedge W)} U + \nabla_{\nabla_{Z \wedge W}(X \wedge Y)} U.
\]
called the first extended curvature and the second extended curvature respectively.

If \(f \in A\) and \(X, Y, Z, W, U \in L\), then we have
\[
f(X \wedge Y) = fX \wedge Y = X \wedge (fY),
\]
\[
R(X, Z \wedge W) Y = -R(Z \wedge W, X) Y,
\]
\[
R(fX, Z \wedge W) Y = R(X, f(Z \wedge W)) Y = fR(X, Z \wedge W) Y,
\]
\[
R(X, Z \wedge W)(fY) = \mathcal{D}^{(1)}(X, Z \wedge W)(f) Y + fR(X, Z \wedge W) Y \quad (4)
\]
and also
\[
R(fX \wedge Y, Z \wedge W) U = R(X \wedge Y, fZ \wedge W) U = fR(X \wedge Y, Z \wedge W) U,
\]
\[
R(X \wedge Y, Z \wedge W) (fU) = \mathcal{D}^{(1)} (X \wedge Y, Z \wedge W) (f) U + fR(X \wedge Y, Z \wedge W) U. \tag{5}
\]

According to their properties, we can consider
\[
R(X, Z \wedge W) (Y \wedge U) \overset{\text{def}}{=} R(X, Z \wedge W) Y \wedge U + Y \wedge R(X, Z \wedge W) U,
\]
\[
R(X \wedge Y, Z \wedge W) (U \wedge V) \overset{\text{def}}{=} R(X \wedge Y, Z \wedge W) U \wedge V + U \wedge R(X \wedge Y, Z \wedge W) V.
\]

**Proposition 3.2** If \( L \) is a preinfinitesimal module and \( X, Y, Z, W, U, V \in L \), then
\[
\mathcal{J}^{(1)} (X, Y, Z \wedge W) = R(Y, Z \wedge W) X - R(X, Z \wedge W) Y,
\]
\[
\mathcal{J}^{(1)} (U, X \wedge Y, Z \wedge W) = R(U, X \wedge Y) (Z \wedge W) - R(U, Z \wedge W) (X \wedge Y) + R(X \wedge Y, Z \wedge W) U,
\]
\[
\mathcal{J}^{(1)} (X \wedge Y, Z \wedge W, U \wedge V) = R(X \wedge Y, Z \wedge W) (U \wedge V) + R(Z \wedge W, U \wedge V) (X \wedge Y) + R(U \wedge V, X \wedge Y) (Z \wedge W).
\]

Proposition 2.2 used for \( L^{(1)} \) gives the following true statement.

**Proposition 3.3** If \( L \) is a preinfinitesimal module, then the following conditions are equivalent:

1. \( L^{(1)} \) is an infinitesimal module;
2. \( \mathcal{D}^{(1)} = 0 \);
3. \( \mathcal{J}^{(1)} \) is an \( A \)–linear form.

*The map \( \mathcal{J}^{(1)} \) vanishes iff \( (L^{(1)}, \rho^{(1)}, [\cdot, \cdot]_{L^{(1)}}) \) is a Lie pseudoalgebra.*

We can explicit condition 2. as follows.

**Proposition 3.4** Let \( L \) be a preinfinitesimal module. The condition that \( L^{(1)} \) is an infinitesimal module is expressed by the relations
\[
[\rho(X), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_X (Z \wedge W)) + \rho(\nabla_{ZA}W) X = 0, \tag{6}
\]
\[
[\mathcal{D}(X, Y), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_{XY} (Z \wedge W) - \nabla_{ZA} (X \wedge Y)) = 0, \tag{7}
\]

(\forall)X, Y, Z, W \in L.

In particular, when \( L \) is an infinitesimal module, we obtain as follows.

**Corollary 3.1** If \( L \) is an infinitesimal module, then \( L^{(1)} \) is an infinitesimal module iff
\[
\rho(R(X, Y) Z) = 0,
\]

(\forall)X, Y, Z \in L.
Proposition 3.5 Consider a preinfinitesimal module \( L \). If \( L^{(1)} \) is an infinitesimal module, then both extended curvatures are \( A \)-linear in their arguments.

We can prove now the main result of the paper.

Theorem 3.1 Let \( L \) be a preinfinitesimal module. If \( L^{(1)} \) is an infinitesimal module and both its extended curvatures vanish, then \( L^{(1)} \) is a Lie pseudoalgebra.

Effectively, the conditions in the hypothesis of the above theorem are expressed by the relations (6), (7) and

\[
R(X,Z \wedge W)Y = R(X \wedge Y,Z \wedge W)U = 0,
\]

\((\forall)X,Y,Z,W \in L.\)

4 Some examples

We consider below two relevant examples. They are the algebraic forms of the two examples presented in [18].

In the first example we consider an infinitesimal module \( L_0 \) that is not a Lie pseudoalgebra, and we prove that its derived preinfinitesimal module \( L_0^{(1)} \) is a Lie pseudoalgebra.

In the second example we consider a preinfinitesimal module \( L_1 \) that is not an infinitesimal module, but its derived preinfinitesimal module \( L_1^{(1)} \) is a Lie pseudoalgebra.

We proceed now with the first example. Let us consider \( A_0 = \mathcal{F}(\mathbb{R}^n) \) the real algebra of smooth functions on \( \mathbb{R}^n \) and the \( A_0 \)-module \( L_0 = \mathcal{M}_n(A_0) \), where \( \mathcal{M}_n(A_0) \) is the set of square \( n \)-matrices with \( A_0 \)-entries. The anchor \( \rho \) on \( L_0 \) is defined in every point \( \bar{x} = (x^1, \ldots, x^n) \) by

\[
\rho\left( X^j_i \right) = \left( x^j \right)^2 \frac{\partial}{\partial x^i}.
\]

The image by \( \rho \) of \( L_0 \) generates the whole tangent space \( T_{\bar{x}}\mathbb{R}^n \) for \( \bar{x} \neq \bar{0} \) and \( \{ \bar{0} \} \subset T_0 \mathbb{R}^n \) for \( \bar{x} = \bar{0} = (0, \ldots, 0) \). A matrix in \( L_0 \) is in \( \ker \rho \) iff it is an \( \mathcal{F}(\mathbb{R}^n) \)-combination of sections \( X_{ijk} = \left( x^j \right)^2 X^k_i - \left( x^k \right)^2 X^j_i \), where \( 1 \leq i < j, k \leq n \). We notice that these \( \frac{n^2(n-1)}{2} \) sections do not generate a (regular) vector subbundle of \( T\mathbb{R}^n \).

Associated with the above anchor, we consider the bracket \([\cdot, \cdot]_{L_0}\) defined on generators by

\[
\left[ X^j_i, X^u_k \right]_{L_0} = 2x^u \delta^i_j X^j_k - 2x^i \delta^u_j X^u_k
\]

and the linear \( L_0 \)-connection \( \nabla \) on \( L_0 \) defined on generators by

\[
\nabla_{X^j_i} X^u_k = 2x^u \delta^i_j X^j_k.
\]

We have \( \rho\left( \left[ X^j_i, X^u_k \right]_{L_0} \right) = \left[ \rho\left( X^j_i \right), \rho\left( X^u_k \right) \right] \), thus \( (L_0, \rho, [\cdot, \cdot]_{L_0}) \) is an infinitesimal module.

The curvature of \( \nabla \) is linear in all arguments and

\[
\nabla_{X^j_i} X^u_k X^v_l = 2 \left( u,v,j,l \right) \left( \left( x^j \right)^2 X^k_l - \left( x^k \right)^2 X^j_l \right) = 2 \left( u,v,j,l \right) X_{kli},
\]

where \( (i,j,k) = \delta^i_j \delta^k_l \) and \( X_{kli} = \left( x^j \right)^2 X^k_l - \left( x^k \right)^2 X^j_l \).

Since \( \rho\left( \nabla_{X^j_i} X^u_k X^v_l \right) = 0 \), then using Corollary 3.1 it follows that the derived bundle \( (L_0^{(1)}, \rho^{(1)}, [\cdot, \cdot]^{(1)}_{L_0}) \) is an infinitesimal module as well. Using Proposition 3.5, it follows that the extended curvatures are \( \mathcal{F}(\mathbb{R}^n) \)-linear in their arguments.
Proposition 4.1 The Jacobiator of $[\cdot, \cdot]_{L_0}$ is

$$\mathcal{J}_{L_0}(X^j_i, X^l_k, X^u_l) = 2(j, u, l) X_{klv} + 2(j, v, k) X_{lul} + 2(i, v, l) X_{ukj}.$$ 

It follows that $L_0$ is not a Lie pseudoalgebra.

Lemma 4.1 For $L_0$, both extended curvatures of $\nabla$ vanish.

Proposition 4.2 The derived preinfinitesimal module $L_0^{(1)}$ is a Lie pseudoalgebra.

We proceed now with the second example. It shows that $L$ is not necessarily a preinfinitesimal module, in the hypothesis of Theorem 3.1.

Consider $M = \mathbb{R}^{2n+1}$ with coordinates $\{x^i, y^i, z_i\}_{i=1}^n$ and the vector fields

$$X_i = \frac{\partial}{\partial x^i} - y^j \frac{\partial}{\partial z^j}, Y_i = \frac{\partial}{\partial y^i}, i = 1, \ldots, n.$$ \hspace{1cm} (10)

Their Lie brackets are given by

$$[X_i, Y_j] = \delta_{ij} \frac{\partial}{\partial z^i}, [X_i, X_j] = [Y_i, Y_j] = 0, i, j = 1, \ldots, n.$$ 

Let $L_1$ be the module generated by $\{X_i, Y_j\}_{i,j=1}^n$. The anchor $\rho_1 : L_1 \to \mathcal{X}(\mathbb{R}^{2n+1})$ is the natural inclusion. The corresponding bracket $[\cdot, \cdot]'$ on $L_1$ is obtained extending the following values on generators:

$$[X_i, Y_j]' = [X_i, X_j]' = [Y_i, Y_j]' = 0, i, j = 1, \ldots, n.$$ 

We consider also the linear $L_1$–connection $\nabla$ on $L_1$, extending the following values on generators:

$$\nabla_{X_i} Y_j = \nabla_{X_i} X_j = \nabla_{Y_i} Y_j = 0, i, j = 1, \ldots, n.$$ 

We have $D'(X_i, Y_j) = [\rho_1(X_i), \rho_1(Y_j)] - \rho_1([X_i, Y_j]) = [X_i, Y_j]' = \delta_{ij} \frac{\partial}{\partial z^i}$, thus $(L_1, \rho_1, [\cdot, \cdot]')$ is not an infinitesimal module and the curvature $R$ is not $\mathcal{F}(\mathbb{R}^{2n+1})$–linear in all arguments.

Proposition 4.3 The derived preinfinitesimal module $L_1^{(1)}$ is a Lie pseudoalgebra.

5 Conclusions

Some relaxed conditions in the Lie pseudoalgebra definition give rise to other kinds of structures (anchored modules, preinfinitesimal modules, infinitesimal modules). For a general preinfinitesimal module, the Jacobiator can be non-null or nonlinear.

A new construction of Lie pseudoalgebras is considered in the paper. A linear connexion $\nabla$ on an anchored module $L$ gives a skew symmetric bracket, thus a preinfinitesimal module structure on $L$ and then on $L^{(1)}$, the derived preinfinitesimal module of $L$. It is proved that the Jacobiator of $L^{(1)}$ can be expressed using the curvature of $\nabla$ and also two extended curvatures of $\nabla$ that are constructed here. In the main result of the paper we prove that if the two extended curvatures vanish, then $L^{(1)}$ is a Lie pseudoalgebra. Two given examples show that the result can be applied not only when $L$ is an infinitesimal module, but also when $L$ is a preinfinitesimal module.
References
