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Extended curvatures and Lie pseudoalgebras

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Abstract. The aim of the paper is to prove that if some extended curvatures on a preinfinitesimal module L, considered in the paper, vanish, then the derived preinfinitesimal module $L^{(1)}$ is a Lie pseudoalgebra. Two non-trivial examples are given. The first example is when L_0 is an infinitesimal module and the second one is when L_1 is a preinfinitesimal module.

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1 Introduction

The Lie pseudoalgebras (see [5, 14, 22]) are generalizations of Lie algebroids [7, 13, 15, 25]. They are known also as Lie-Rinehart algebras (see, for example, [12]). Some less conditions in the definition of a Lie pseudoalgebra gives more general structures, known under different names and using different forms, but we use the most definitions in [21, 22, 25]. A general 'definition gives rise to anchored modules, that correspond to anchored vector bundles considered in [19], as relative tangent spaces; the infinitesimal modules correspond, via vector bundles setting, to skew symmetric Lie algebroids [17]; the preinfinitesimal modules correspond on vector bundles to almost Lie vector bundles, called as almost Lie structures in [16, 20, 23].

In [22] we define the (first) derived preinfinitesimal module of a given preinfinitesimal module, and we use it in the present paper. We suppose that a linear L–connexion ∇ with a null torsion exists, using it in the background. Two extended curvatures of ∇ are defined, and we prove that a sufficient condition that the derived preinfinitesimal module $L^{(1)}$ be a Lie pseudoalgebra is that both these two extended curvatures vanish (the main Theorem 3.1).

The main Theorem 3.1 is applied for two examples. In the first example E_0 is an infinitesimal module and the second one is when E_1 is not an infinitesimal module. The results are algebraic versions of vector bundle constructions, using Lie algebroids and skew-symmetric vector bundles [18].

2 Preinfinitesimal modules

Consider a commutative ring $\mathbf{k}$ and and a commutative and associative $\mathbf{k}$-algebra A. When L is an A–module, we say also that (A, L) is a module. For example, we have the module of

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derivations on A, $(A, \mathcal{D}er(A))$, where $\mathcal{D}er(A)$ is the $\mathbf{k}$–module of maps $X : A \to A$, such that $X(a \cdot b) = X(a) \cdot b + a \cdot X(b)$.

A couple (L, ρ) is an *anchored module* if (A, L) is a module and $\rho : L \to \mathcal{D}er(A)$ an A-linear map, called an *anchor* (or an *arrow*). A triple $(L, \rho, [Y, X]_L)$ is a *preinfinitesimal module* if (L, ρ) is an anchored module and $[\cdot, \cdot]_L$ is a *bracket*, i.e. a $\mathbf{k}$-bilinear map $[\cdot, \cdot]_L : L \times L \to L$ which is skew symmetric, i.e.

$$[Y, X]_L = -[X, Y]_L,$$

$(\forall)X, Y \in L$ and fulfills the property

$$[X, a \cdot Y]_L = \rho(X)(a) \cdot Y + a \cdot [X, Y]_L , (\forall)X, Y \in L, a \in A.$$

Consider now the maps

$$\mathcal{D} : L \times L \overset{not}{=} L^2 \to Der(A), \mathcal{D}(X, Y) = [\rho(X), \rho(Y)] - \rho([X, Y]_L)$$

and

$$\mathcal{J}_L : L^3 \to L, \mathcal{J}_L(X, Y, Z) = \sum_{circ} [X, [Y, Z]_L]_L.$$

As for example in [6]) $\mathcal{J}_L$ is called the *Jacobiator* of the bracket $[\cdot, \cdot]_L$.

In general, on a preinfinitesimal module L, $\mathcal{D}$ is A-linear in all its arguments, while $\mathcal{J}_L$ is not.

If the compatibility condition for brackets, $\mathcal{D} = 0$, holds, then L is called an *infinitesimal module*.

An infinitesimal module is called a *Lie pseudoalgebra* if the Jacobiator of the bracket vanishes.

For sake of simplicity, we denote simply by L a structure on L.

All their properties can be summarized as follows. The proofs are straightforward and follow by direct computations.

Proposition 2.1 *Let L be a preinfinitesimal module, with a skew-symmetric bracket. Then the followifng statements hold true:*

1. The map $\mathcal{D}$ is A–bilinear and skew symmetric.

2. The map $\mathcal{J}_L$ is skew symmetric and $\mathcal{J}_L(X, Y, fZ) = \mathcal{D}(X, Y)(f)Z + f\mathcal{J}_L(X, Y, Z)$, $(\forall)$ $f \in A$ and $X, Y, Z \in L$.

3. The map $\mathcal{J}_L$ is A–linear in all its arguments iff L is an infinitesimal module.

4. The map $\mathcal{J}_L$ vanishes iff L is a Lie pseudoalgebra.

In the particular case of infinitesimal modules we obtain as follows.

Proposition 2.2 *For a preinfinitesimal module L with a skew-symmetric bracket, the following conditions are equivalent:*

1. *L is an infinitesimal module;*

2. $\mathcal{D} = 0$*;*

3. *$\mathcal{J}_L$ is A–linear in all its three arguments.*

The Jacobiator $\mathcal{J}_L$ vanishes iff L is a Lie pseudoalgebra.

The map $\mathcal{D}$ can be seen as well as an anchor $\rho^{\wedge} : L \wedge L \to Der\,(A)$,

$$\rho^{\wedge}\,(X \wedge Y) = \mathcal{D}\,(X, Y)\,,$$

where $L \wedge L$ denotes the exterior two product module of L over A.

Consider now an anchored module L and let (A, M) be a module. A *linear L-connection* on M is a map $\nabla : L \times M \to M$ that verifies Koszul conditions:

$$\nabla_{fX} s = f \nabla_X s, \nabla_{(X+X')} s = \nabla_X s + \nabla_{X'} s,$$
$$\nabla_X\,(fs) = \rho\,(X)\,(f)\,s + f \nabla_X s, \nabla_X\,(s + s') = \nabla_X s + \nabla_X s',$$

$(\forall) X, X' \in L$, $s, s' \in M$, $f \in A$.

We suppose that such a linear L–connection ∇ exists.

Consider now that L is a preinfinitesimal module. The *curvature* of ∇ is the map $R : L^2 \times M \to M$, given by the formula

$$R\,(X, Y)\,s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]_L} s,$$

$(\forall) X, Y \in L$ and $s \in M$.

Proposition 2.3 *Let L be a preinfinitesimal module,* (A, M) *be a module and* ∇ *be a linear L-connection on M. Then the formula* $R_{X \wedge Y} s = R\,(X, Y)\,s$ *defines a linear* $L \wedge L$*–connection on M, according to the anchor* $\rho^{\wedge}$*.*

Now let us consider the particular case of an L–connection ∇ on an anchored module L. In this case $[\cdot, \cdot]_L : L \times L \to L$, $[X, Y]_L \overset{def.}{=} \nabla_X Y - \nabla_Y X$, is a bracket on L, so L is a preinfinitesimal module.

When L is a preinfinitesimal module, we can consider the *torsion* of an L–connection ∇ as a map

$$T : L^2 \to L,$$

given by

$$T\,(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_L.$$

The curvature of ∇ is $R : L^3 \to L$, given by

$$R\,(X, Y)\,Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]_L} Z \overset{not.}{=} \nabla_{X \wedge Y} Z,$$

$(\forall) X, Y, Z \in L$. Then R defines a linear $L \wedge L$–connection $\nabla^{\wedge}$ on L, according to the anchor $\rho^{\wedge}$:

$$\nabla^{\wedge}_{X \wedge Y} Z \overset{def.}{=} \nabla_{X \wedge Y} Z.$$

Using the formula

$$\nabla^{(\wedge)}_{X \wedge Y}\,(Z \wedge W) = \nabla_{X \wedge Y} Z \wedge W + Z \wedge \nabla_{X \wedge Y} W \overset{not.}{=} \nabla_{X \wedge Y}\,(Z \wedge W)$$

we get a linear $L \wedge L$–connection $\nabla^{(\wedge)}$ on $L \wedge L$, according to the anchor $\rho^{\wedge}$.

Notice that when the torsion of ∇ vanishes, then

$$[X, Y]_L = \nabla_X Y - \nabla_Y X. \qquad (1)$$

3 The derived preinfinitesimal module

Let (ρ, L) be an anchored module such that there is an L–connexion ∇ on L. Thus L is a preinfinitesimal module and ∇ has a null torsion.

On the module $L^{(1)} = L \oplus (L \wedge L)$ there is an anchor given by the formula

$$\rho^{(1)}(X + (Y \wedge Z)) = \rho(X) + \rho^{\wedge}(Y \wedge Z)$$

and an $L^{(1)}$–connection $\nabla^{(1)}$ on $L^{(1)}$, according to the formulas

$$\nabla^{(1)}_X Y = \nabla_X Y + \frac{1}{2} X \wedge Y,\ \nabla^{(1)}_X (Y \wedge Z) = \nabla_X (Y \wedge Z) = \nabla_X Y \wedge Z + Y \wedge \nabla_X Z, \tag{2}$$
$$\nabla^{(1)}_{X \wedge Y} Z = \nabla_{X \wedge Y} Z,\ \nabla^{(1)}_{X \wedge Y} (Z \wedge T) = \nabla_{X \wedge Y} (Z \wedge T).$$

A bracket on $L^{(1)}$ according to the anchor $\rho^{(1)}$ can be given by

$$[U, V]_{L^{(1)}} = \nabla^{(1)}_U V - \nabla^{(1)}_V U.$$

More exactly, the bracket on $L^{(1)}$ is given explicitly by the formulas

$$\begin{aligned} [X, Y]_{L^{(1)}} &= [X, Y]_L + X \wedge Y, \\ [X \wedge Y, Z]_{L^{(1)}} &= \nabla_{X \wedge Y} Z - \nabla_Z (X \wedge Y), \\ [X \wedge Y, Z \wedge T]_{L^{(1)}} &= \nabla_{X \wedge Y} (Z \wedge T) - \nabla_{Z \wedge T} (X \wedge Y). \end{aligned} \tag{3}$$

According to [22], the preinfinitesimal module $(L^{(1)}, \rho^{(1)}, [\cdot, \cdot]_{L^{(1)}})$ is called the (*first*) *derived preinfinitesimal module* of (L, ρ), given by ∇.

Proposition 3.1 *Let us 'ose the linear L-connection* ∇ *on the preinfinitesimal module L has no torsion. Then the following properties hold true:*

1. $\mathcal{D}^{(1)}(X, Y) = [\rho^{(1)}(X), \rho^{(1)}(Y)] - \rho^{(1)}([X, Y]_{\theta^{(1)}}) = 0$, $(\forall) X, Y \in L$.
2. $\mathcal{J}^{(1)}(X, Y, Z) = 0$, $(\forall) X, Y, Z \in L$, *where* $\mathcal{J}^{(1)}$ *is the Jacobiator of* $[\cdot, \cdot]_{L^{(1)}}$.

We are going to show that the identical non-vanishing components of the Jacobiator on $L^{(1)}$ can be expressed using two other curvatures involving ∇, that we define below.

For this, let us consider

$$(X, Y, Z, W) \rightarrow R(X, Z \wedge W) Y \overset{def.}{=} \nabla_X \nabla_{Z \wedge W} Y - \nabla_{Z \wedge W} \nabla_X Y - \nabla_{\nabla_X (Z \wedge W)} Y + \nabla_{\nabla_{Z \wedge W} X} Y,$$
$$R(Z \wedge W, X) Y \overset{def.}{=} -R(X, Z \wedge W) Y$$

and

$$(X, Y, Z, W, U) \rightarrow R(X \wedge Y, Z \wedge W) U \overset{def.}{=} \nabla_{X \wedge Y} \nabla_{Z \wedge W} U - \nabla_{Z \wedge W} \nabla_{X \wedge Y} U -$$
$$\nabla_{\nabla_{X \wedge Y}(Z \wedge W)} U + \nabla_{\nabla_{Z \wedge W}(X \wedge Y)} U.$$

called the *first extended curvature* and the *second extended curvature* respectively.

If $f \in A$ and $X, Y, Z, W, U \in L$, then we have

$$\begin{aligned} f(X \wedge Y) &= fX \wedge Y = X \wedge (fY), \\ R(X, Z \wedge W) Y &= -R(Z \wedge W, X) Y, \\ R(fX, Z \wedge W) Y &= R(X, f(Z \wedge W)) Y = fR(X, Z \wedge W) Y, \\ R(X, Z \wedge W)(fY) &= \mathcal{D}^{(1)}(X, Z \wedge W)(f) Y + fR(X, Z \wedge W) Y \end{aligned} \tag{4}$$

and also

$$R(fX \wedge Y, Z \wedge W)U = R(X \wedge Y, fZ \wedge W)U = fR(X \wedge Y, Z \wedge W)U,$$
$$R(X \wedge Y, Z \wedge W)(fU) = \mathcal{D}^{(1)}(X \wedge Y, Z \wedge W)(f)U + fR(X \wedge Y, Z \wedge W)U. \tag{5}$$

According to their properties, we can consider

$$R(X, Z \wedge W)(Y \wedge U) \overset{def.}{=} R(X, Z \wedge W)Y \wedge U + Y \wedge R(X, Z \wedge W)U,$$
$$R(X \wedge Y, Z \wedge W)(U \wedge V) \overset{def.}{=} R(X \wedge Y, Z \wedge W)U \wedge V+$$
$$U \wedge R(X \wedge Y, Z \wedge W)V.$$

Proposition 3.2 *If L is a preinfinitesimal module and $X, Y, Z, W, U, V \in L$, then*

$$\mathcal{J}^{(1)}(X, Y, Z \wedge W) = R(Y, Z \wedge W)X - R(X, Z \wedge W)Y,$$
$$\mathcal{J}^{(1)}(U, X \wedge Y, Z \wedge W) = R(U, X \wedge Y)(Z \wedge W) - R(U, Z \wedge W)(X \wedge Y) + R(X \wedge Y, Z \wedge W)U,$$
$$\mathcal{J}^{(1)}(X \wedge Y, Z \wedge W, U \wedge V) = R(X \wedge Y, Z \wedge W)(U \wedge V) + R(Z \wedge W, U \wedge V)(X \wedge Y) + R(U \wedge V, X \wedge Y)(Z \wedge W).$$

Proposition 2.2 used for $L^{(1)}$ gives the following true statement.

Proposition 3.3 *If L is a preinfinitesimal module, then the following conditions are equivalent:*

1. *$L^{(1)}$ is an infinitesimal module;*
2. *$\mathcal{D}^{(1)} = 0$;*
3. *$\mathcal{J}^{(1)}$ is an A–linear form.*

The map $\mathcal{J}^{(1)}$ vanishes iff $(L^{(1)}, \rho^{(1)}, [\cdot,\cdot]_{L^{(1)}})$ is a Lie pseudoalgebra.

We can explicit condition 2. as follows.

Proposition 3.4 *Let L be a preinfinitesimal module. The condition that $L^{(1)}$ is an infinitesimal module is expressed by the relations*

$$[\rho(X), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_X(Z \wedge W)) + \rho(\nabla_{Z \wedge W}X) = 0, \tag{6}$$
$$[\mathcal{D}(X, Y), \mathcal{D}(Z, W)] - \mathcal{D}(\nabla_{X \wedge Y}(Z \wedge W) - \nabla_{Z \wedge W}(X \wedge Y)) = 0, \tag{7}$$

$(\forall)X, Y, Z, W \in L.$

In particular, when L is an infinitesimal module, we obtain as follows.

Corollary 3.1 *If L is an infinitesimal module, then $L^{(1)}$ is an infinitesimal module iff*

$$\rho(R(X, Y)Z) = 0,$$

$(\forall)X, Y, Z \in L.$

Proposition 3.5 *Consider a preinfinitesimal module L. If $L^{(1)}$ is an infinitesimal module, then both extended curvatures are A–linear in their arguments.*

We can prove now the main result of the paper.

Theorem 3.1 *Let L be a preinfinitesimal module. If $L^{(1)}$ is an infinitesimal module and both its extended curvatures vanish, then $L^{(1)}$ is a Lie pseudoalgebra.*

Effectively, the conditions in the hypothesis of the above theorem are expressed by the relations (6), (7) and

$$R(X, Z \wedge W)Y = R(X \wedge Y, Z \wedge W)U = 0, \tag{8}$$

$(\forall) X, Y, Z, W \in L$.

4 Some examples

We consider below two relevant examples. They are the algebraic forms of the two examples presented in [18].

In the first example we consider an infinitesimal module L_0 that is not a Lie pseudoalgebra, and we prove that its derived preinfinitesimal module $L_0^{(1)}$ is a Lie pseudoalgebra.

In the second example we consider a preinfinitesimal module L_1 that is not an infinitesimal module, but its derived preinfinitesimal module $L_1^{(1)}$ is a Lie pseudoalgebra.

We proceed now with the first example. Let us consider $A_0 = \mathcal{F}(\mathbb{R}^n)$ the real algebra of smooth functions on $\mathbb{R}^n$ and the A_0–module $L_0 = \mathcal{M}_n(A_0)$, where $\mathcal{M}_n(A_0)$ is the set of square n–matrices with A_0–entries. The anchor ρ on L_0 is defined in every point $\bar{x} = \left(x^1, \ldots, x^n\right)$ by

$$\rho\left(X_i^j\right) = \left(x^j\right)^2 \frac{\partial}{\partial x^j}. \tag{9}$$

The image by ρ of L_0 generates the whole tangent space $T_{\bar{x}}\mathbb{R}^n$ for $\bar{x} \neq \bar{0}$ and $\{\bar{0}\} \subset T_{\bar{0}}\boldsymbol{R}^n$ for $\bar{x} = \bar{0} = (0, \ldots, 0)$. A matrix in L_0 is in $\ker\rho$ iff it is an $\mathcal{F}(\boldsymbol{R}^n)$-combination of sections $X_{ijk} = \left(x^j\right)^2 X_k^i - \left(x^i\right)^2 X_k^j$, where $1 \leq i < j, k \leq n$. We notice that these $\frac{n^2(n-1)}{2}$ sections do not generate a (regular) vector subbundle of $T\mathbb{R}^n$.

Associated with the above anchor, we consider the bracket $[\cdot,\cdot]_{L_0}$ defined on generators by

$$\left[X_j^i, X_v^u\right]_{L_0} = 2x^u\delta_j^u X_v^i - 2x^i\delta_v^i X_j^u$$

and the linear L_0–connection ∇ on L_0 defined on generators by

$$\nabla_{X_j^i} X_v^u = 2x^u\delta_j^u X_v^i.$$

We have $\rho\left(\left[X_j^i, X_l^k\right]_{L_0}\right) = \left[\rho\left(X_j^i\right), \rho\left(X_l^k\right)\right]$, thus $\left(L_0, \rho, [\cdot,\cdot]_{L_0}\right)$ is an infinitesimal module.

The curvature of ∇ is linear in all arguments and

$$\nabla_{X_j^i \wedge X_l^k} X_v^u = 2(u, j, l)\left(\left(x^i\right)^2 X_v^k - \left(x^k\right)^2 X_v^i\right) = 2(u, j, l)\, X_{kiv},$$

where $(i, j, k) = \delta_j^i \delta_k^i$ and $X_{kiv} = \left(x^i\right)^2 X_v^k - \left(x^k\right)^2 X_v^i$.

Since $\rho\left(\nabla_{X_j^i \wedge X_l^k} X_v^u\right) = 0$, then using Corollary 3.1 it follows that the derived bundle $\left(L_0^{(1)}, \rho^{(1)}, [\cdot,\cdot]_{L_0}^{(1)}\right)$ is an infinitesimal module as well. Using Proposition 3.5, it follows that the extended curvatures are $\mathcal{F}(\mathbb{R}^n)$–linear in their arguments.

Proposition 4.1 *The Jacobiator of* $[\cdot,\cdot]_{L_0}$ *is*

$$\mathcal{J}_{L_0}\left(X^i_j, X^k_l, X^u_v\right) = 2\,(j,u,l)\,X_{kiv} + 2\,(j,v,k)\,X_{iul} + 2\,(i,v,l)\,X_{ukj}.$$

It follows that L_0 is not a Lie pseudoalgebra.

Lemma 4.1 *For* L_0*, both extended curvatures of* ∇ *vanish.*

Proposition 4.2 *The derived preinfinitesimal module* $L_0^{(1)}$ *is a Lie pseudoalgebra.*

We proceed now with the second example. It shows that L is not necessarily a preinfinitesimal module, in the hypothesis of Theorem 3.1.

Consider $M = \mathbb{R}^{2n+1}$ with coordinates $\left\{x^i, y^i, z\right\}_{i=\overline{1,n}}$ and the vector fields

$$X_i = \frac{\partial}{\partial x^i} - y^i\frac{\partial}{\partial z}, Y_i = \frac{\partial}{\partial y^i}, i = \overline{1,n}. \tag{10}$$

Their Lie brackets are given by

$$\left[X_i, Y_j\right] = \delta_{ij}\frac{\partial}{\partial z}, \left[X_i, X_j\right] = \left[Y_i, Y_j\right] = 0, i, j = \overline{1,n}.$$

Let L_1 be the module generated by $\left\{X_i, Y_j\right\}_{i,j=\overline{1,n}}$. The anchor $\rho_1 : L_1 \to \mathcal{X}\left(\mathbb{R}^{2n+1}\right)$ is the natural inclusion. The corresponding bracket $[\cdot,\cdot]'$ on L_1 is obtained extending the following values on generators:

$$\left[X_i, Y_j\right]' = \left[X_i, X_j\right]' = \left[Y_i, Y_j\right]' = 0, i, j = \overline{1,n}.$$

We consider also the linear L_1–connection ∇ on L_1, extending the following values on generators:

$$\nabla_{X_i}Y_j = \nabla_{X_i}X_j = \nabla_{Y_i}Y_j = 0, i, j = \overline{1,n}.$$

We have $\mathcal{D}'\left(X_i, Y_j\right) = \left[\rho_1\left(X_i\right), \rho_1\left(Y_j\right)\right] - \rho_1\left(\left[X_i, Y_j\right]'\right) = \left[X_i, Y_j\right] = \delta_{ij}\frac{\partial}{\partial z}$, thus $(L_1, \rho_1, [\cdot,\cdot]')$ is not an infinitesimal module and the curvature R is not $\mathcal{F}\left(\mathbb{R}^{2n+1}\right)$–linear in all arguments.

Proposition 4.3 *The derived preinfinitesimal module* $L_1^{(1)}$ *is a Lie pseudoalgebra.*

5 Conclusions

Some relaxed conditions in the Lie pseudoalgebra definition give rise to other kinds of structures (anchored modules, preinfinitesimal modules, infinitesimal modules). For a general preinfinitesimal module, the Jacobiator can be non-null or nonlinear.

A new construction of Lie pseudoalgebras is considered in the paper. A linear connexion ∇ on an anchored module L gives a skew symmetric bracket, thus a preinfinitesimal module structure on L and then on $L^{(1)}$, the derived preinfinitesimal module of L. It is proved that the Jacobiator of $L^{(1)}$ can be expressed using the curvature of ∇ and also two extended curvatures of ∇ that are constructed here. In the main result of the paper we prove that if the two extended curvatures vanish, then $L^{(1)}$ is a Lie pseudoalgebra. Two given examples show that the result can be applied not only when L is an infinitesimal module, but also when L is a preinfinitesimal module.

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