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Defining the speed independence of the Boolean asynchronous systems

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Abstract. A discrete time Boolean asynchronous system consists in a function $\Phi : \{0, 1\}^n \to \{0, 1\}^n$ which iterates its coordinates $\Phi_1, ..., \Phi_n$ independently of each other. The durations of computation of $\Phi_1, ..., \Phi_n$ are supposed to be unknown. The analysis of such systems has as main challenge characterizing their dynamics in conditions of uncertainty. For this, a very cited classical paper is [1], where the fundamental concept of speed independence is introduced. The point is, like in most of these cases, that the engineers receive from such a work intuition, combined with a certain lack of rigor. Our aim is to try a mathematical reinforcement of the Muller's theory of the asynchronous circuits, which should be a modest homage, over time, to its authors.
A list of models used in asynchronous systems theory is given in [3]. The mathematical tools used in this analysis may be found in [2].

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1 Preliminaries

Definition 1.1 *The* ***binary Boole*** *(or* ***Boolean****)* ***algebra*** *is the set* $\mathbf{B} = \{0, 1\}$ *endowed with the following laws, see Table 1: '—' is called (****logical****)* ***complement****, ' · ' is the* ***product****, and*

Table 1. The laws of **B**.

—		·	0	1	∪	0	1
0	1	0	0	0	0	0	1
1	0	1	0	1	1	1	1

'∪' is the ***union****. These laws induce on* $\mathbf{B}^n$ *laws which act coordinatewisely, and have the same notations.*

Definition 1.2 *For* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *and* $\lambda \in \mathbf{B}^n$*, we define the function* $\Phi^\lambda : \mathbf{B}^n \to \mathbf{B}^n$*, called the* λ*–****iterate*** *of* Φ*, by* $\forall \mu \in \mathbf{B}^n, \forall i \in \{1, ..., n\}$,

$$\Phi_i^\lambda(\mu) = \begin{cases} \mu_i, \ if \ \lambda_i = 0, \\ \Phi_i(\mu), \ if \ \lambda_i = 1. \end{cases}$$

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Definition 1.3 *The function $\alpha : \mathbf{N} \longrightarrow \mathbf{B}^n$, $\mathbf{N} \ni k \to \alpha^k \in \mathbf{B}^n$ is called* ***computation function****. $\mathbf{N}$ is the* ***time set****, and k is the* ***time (instant)****. If one of the equivalent properties*[1]

$$\forall i \in \{1, ..., n\},\ \text{the sets } \{k | k \in \mathbf{N}, \alpha_i^k = 1\} \text{ are infinite,}$$

$$\forall k \in \mathbf{N}, \exists k' > k, \alpha^k \cup ... \cup \alpha^{k'} = (1, ..., 1)$$

is true, α is said to be ***progressive****. The set of the progressive computation functions is denoted with Π_n.*

Remark 1.1 *The $\lambda-$iterate Φ^λ shows how the function Φ is computed: Φ^λ computes only these coordinates Φ_i, $i \in \{1, ..., n\}$ for which $\lambda_i = 1$, and the rest of the coordinates keep their values. These are the asynchronous computations of the Boolean functions, considered timelessly.*

The timeful asynchronous computation of Φ makes use of α^k-iterates Φ^{α^k}, showing how and when (due to $k \in \mathbf{N}$) Φ is computed. The requirement of progressiveness of α refers to the progress of time. One possible interpretation of the statement $\alpha_i^k = 1$ is: time advances on the i−th coordinate with 1 time unit.

Remark 1.2 *If $\alpha : \mathbf{N} \to \mathbf{B}^n$ is periodic: $\exists p \geq 1, \forall k \in \mathbf{N}$,*

$$\alpha^k = \alpha^{k+p}, \tag{1}$$

then its progressiveness is equivalent with $\forall k \in \mathbf{N}$,

$$\alpha^k \cup ... \cup \alpha^{k+p-1} = (1, ..., 1). \tag{2}$$

The limit situation in this statement is represented by $p = 1$ and the progressive computation function $\forall k \in \mathbf{N}, \alpha^k = (1, ..., 1)$. If we replace the periodicity of α with the more general property of eventual periodicity, which is: $\exists p \geq 1, \exists k' \in \mathbf{N}, \forall k \geq k'$, (1) holds, then the progressiveness of α is equivalent with: $\forall k \geq k'$, (2) is true.

Definition 1.4 *We consider the function $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, the progressive computation function $\alpha \in \Pi_n$ and $\mu \in \mathbf{B}^n$. The function $\forall k \in \mathbf{N}$,*

$$\phi^\alpha(\mu, k) = \begin{cases} \mu, \ if\ k = 0, \\ \Phi^{\alpha^{k-1}}(\phi^\alpha(\mu, k-1)), \ if\ k \geq 1 \end{cases}$$

is called ***flow****. In this context $\mathbf{B}^n$ is called* ***state space*** *and its elements are called* ***states****, function Φ is called* ***system,*** *or* ***generator function (of*** *ϕ), $x : \mathbf{N} \longrightarrow \mathbf{B}^n$ given by*

$$x(k) = \phi^\alpha(\mu, k)$$

is called ***state function,*** *and μ is the* ***initial value*** *of x, or the* ***initial state****.*

Definition 1.5 *For any $k' \in \mathbf{N}$, the* ***forgetful function*** *$\sigma^{k'} : (\mathbf{B}^n)^{\mathbf{N}} \to (\mathbf{B}^n)^{\mathbf{N}}$ is defined as $\forall x : \mathbf{N} \to \mathbf{B}^n, \forall k \in \mathbf{N}$,*

$$\sigma^{k'}(x)(k) = x(k + k').$$

Remark 1.3 *$\sigma^{k'}$ shifts the function $x : \mathbf{N} \to \mathbf{B}^n$ with k' time units. Its name comes from the fact that for any $k' \geq 1$, $\sigma^{k'}(x)$ forgets the first values $x(0), ..., x(k'-1)$ of x.*

Theorem 1.1 *(Composition) $\forall \alpha \in \Pi_n, \forall \mu \in \mathbf{B}^n, \forall \mu' \in \mathbf{B}^n, \forall k' \in \mathbf{N}$,*

$$\phi^\alpha(\mu, k') = \mu' \Longrightarrow \forall k \in \mathbf{N}, \phi^\alpha(\mu, k + k') = \phi^{\sigma^{k'}(\alpha)}(\mu', k). \tag{3}$$

[1] The proof of the equivalence of these properties is omitted.

Proof. We suppose that $\phi^\alpha(\mu, k') = \mu'$ and we use the induction on $k \in \mathbf{N}$. For $k = 0$ the equality holds, thus we can suppose that it is true for k. We get

$$\phi^\alpha(\mu, k + k' + 1) = \Phi^{\alpha^{k+k'}}(\phi^\alpha(\mu, k + k')) = \Phi^{\alpha^{k+k'}}(\phi^{\sigma^{k'}(\alpha)}(\mu', k))$$

$$= \Phi^{(\sigma^{k'}(\alpha))^k}(\phi^{\sigma^{k'}(\alpha)}(\mu', k)) = \phi^{\sigma^{k'}(\alpha)}(\mu', k + 1).$$

■

Remark 1.4 *Equivalently, (3) can be written as:*

$$\sigma^{k'}(\phi^\alpha(\mu, \cdot))(k) = \phi^\alpha(\mu, k + k') = \phi^{\sigma^{k'}(\alpha)}(\phi^\alpha(\mu, k'), k), \tag{4}$$

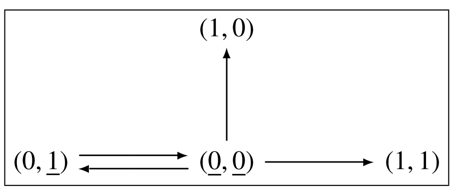
with arbitrary $\alpha \in \Pi_n$, $\mu \in \mathbf{B}^n$, $k \in \mathbf{N}$, *and* $k' \in \mathbf{N}$.

Definition 1.6 *The set*

$$O^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \in \mathbf{N}\}$$

*is called **orbit**,* $\mu \in \mathbf{B}^n$ *is the **initial value** of the orbit and* $\alpha \in \Pi_n$ *is its **computation function**.*

Example 1.1 *Timelessly, the dynamics of these systems is described by directed graphs called **state portraits**. In this example, the system* $\Phi : \mathbf{B}^2 \to \mathbf{B}^2, \Phi(0,0) = \Phi(1,1) = (1,1), \Phi(0,1) = (0,0), \Phi(1,0) = (1,0)$



starts from the initial value $(0,0)$. *In the state portrait, we underline* $\underline{\mu_i}$ *the coordinates* $i \in \{1,2\}$ *so called unstable (or excited) which, by computation, change their value. The arrows indicate the transfer from one state to the other. If* $\Phi_1(0,0)$ *is computed first, the transfer* $(0,0) \to (1,0)$ *takes place and the system remains in* $(1,0)$, *which is a rest position. And if* $\Phi_1(0,0), \Phi_2(0,0)$ *are computed at the same time, the transfer* $(0,0) \to (1,1)$ *takes place and the system remains in* $(1,1)$, *which is a rest position too. If* $\Phi_2(0,0)$ *is computed first the transfer* $(0,0) \to (0,1)$ *takes place and the possibility exists that the system switches between* $(0,0)$ *and* $(0,1)$ *infinitely many times or perhaps, after finitely many such switches, that it eventually reaches one of the rest positions* $(1,0)$ *or* $(1,1)$. *The durations of computation of* Φ_1, Φ_2 *are unknown, meaning that all these transfers are possible, in other words the timeful analysis of the system is made by considering* α *as parameter.*

2 Invariance

Theorem 2.1 *The system* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *and the set* $A \subset \mathbf{B}^n, A \neq \emptyset$ *are considered. The following statements are equivalent:*

$$\forall \alpha \in \Pi_n, \forall \mu \in A, O^\alpha(\mu) \subset A, \tag{5}$$

$$\forall \lambda \in \mathbf{B}^n, \Phi^\lambda(A) \subset A. \tag{6}$$

Proof. (5)$\Longrightarrow$(6) Let $\lambda \in \mathbf{B}^n, \mu \in A$ arbitrary, fixed. We take $\alpha \in \Pi_n$ arbitrary, with $\alpha^0 = \lambda$. Then:

$$\Phi^\lambda(\mu) = \phi^\alpha(\mu, 1) \overset{(5)}{\in} A.$$

(6)$\Longrightarrow$(5) We take $\alpha \in \Pi_n, \mu \in A$ arbitrary, fixed and we prove (5) by induction on k. For $k = 0, \mu = \phi^\alpha(\mu, 0) \in A$ and we suppose now that $\phi^\alpha(\mu, k) \in A$. Then:

$$\phi^\alpha(\mu, k+1) = \Phi^{\alpha^k}(\phi^\alpha(\mu, k)) \overset{(6)}{\in} A.$$

■

Definition 2.1 *If the set A fulfills one of (5), (6), then it is called* ***invariant****.*

Remark 2.1 *Invariance states, in a timeful way, and also in an equivalent timeless way, that any orbit with the initial value in A remains in A.*

Example 2.1 *We look again at the state portrait from Example 1.1. We note there the invariant sets* $\{(1,0)\}, \{(1,1)\}$ *and* $\mathbf{B}^2$.

3 Omega limit sets

Notation 3.1 *The system* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, $\alpha \in \Pi_n$ *and* $\mu \in \mathbf{B}^n$ *are given. We denote with* $\omega_p^\alpha(\mu) \subset \mathbf{B}^n, p \in \mathbf{N}$ *the sets*

$$\omega_p^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \geq p\}.$$

Theorem 3.1 *We have*

$$O^\alpha(\mu) = \omega_0^\alpha(\mu) \supset \omega_1^\alpha(\mu) \supset ... \supset \omega_p^\alpha(\mu) \supset \omega_{p+1}^\alpha(\mu) \supset ...$$

and $k' \in \mathbf{N}$ *exists with the property* $\omega_{k'}^\alpha(\mu) = \omega_{k'+1}^\alpha(\mu) = ...$

Proof. The inclusions are obvious and the property results from the fact that there are finitely many subsets of $O^\alpha(\mu)$. ■

Definition 3.1 *The set* $\omega_{k'}^\alpha(\mu)$ *from the previous theorem is denoted* $\omega^\alpha(\mu)$ *and is called* ***omega limit****, or* ***terminal****. In general, a set* $A \subset \mathbf{B}^n, A \neq \emptyset$ *is called omega limit or terminal if* $\alpha \in \Pi_n$ *and* $\mu \in \mathbf{B}^n$ *exist with the property that* $A = \omega^\alpha(\mu)$.

Notation 3.2 *The set of the omega limit sets of* Φ *is denoted* Ω_Φ :

$$\Omega_\Phi = \{\omega^\alpha(\mu) | \alpha \in \Pi_n, \mu \in \mathbf{B}^n\}.$$

Theorem 3.2 *Let* $\alpha \in \Pi_n$ *and* $\mu \in \mathbf{B}^n$.
(a) $k' \in \mathbf{N}$ *exists such that*

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \geq k'\}. \tag{7}$$

(b) We have

$$\omega^\alpha(\mu) = \{\nu | \nu \in \mathbf{B}^n, \textit{ the set } \{k | k \in \mathbf{N}, \phi^\alpha(\mu, k) = \nu\} \textit{ is infinite}\}.$$

(c) If k' *satisfies either of* $\forall k_1 \geq k', \forall k_2 \geq k'$,

$$\{\phi^\alpha(\mu, k) | k \geq k_1\} = \{\phi^\alpha(\mu, k) | k \geq k_2\},$$

respectively $\forall k_1 \geq k'$,

$$\{\phi^\alpha(\mu, k) | k \geq k_1\} = \{\phi^\alpha(\mu, k) | k \geq k'\},$$

then (7) is true.

Proof. (a) and (c) follow directly from Definition 3.1, we prove (b). We know that $\omega^{\alpha}(\mu) \subset O^{\alpha}(\mu)$. On the other hand the supposition that we might have $\nu \in \omega^{\alpha}(\mu)$ with the set $\{k|k \in \mathbf{N}, \phi^{\alpha}(\mu, k) = \nu\}$ finite brings the contradiction $\forall k'' > \max\{k|k \in \mathbf{N}, \phi^{\alpha}(\mu, k) = \nu\}$,

$$\omega^{\alpha}(\mu) \supsetneq \omega^{\alpha}_{k''}(\mu).$$

■

Remark 3.1 *We conclude that the omega limit set is the nonempty subset of the orbit* $\omega^{\alpha}(\mu) \subset O^{\alpha}(\mu)$ *which contains the points reached by the state* $x(k) = \phi^{\alpha}(\mu, k)$ *infinitely many times. In particular, the periodicity of the state function:* $\exists p \geq 1, \forall k \in \mathbf{N}$,

$$\phi^{\alpha}(\mu, k) = \phi^{\alpha}(\mu, k + p), \tag{8}$$

when all its values are reached infinitely many times, implies $O^{\alpha}(\mu) = \omega^{\alpha}(\mu)$. *And in case of eventual periodicity:* $\exists p \geq 1, \exists k' \in \mathbf{N}, \forall k \geq k'$, *(8) is true, we get* $O^{\alpha}(\mu) \supset \omega^{\alpha}_{k'}(\mu) = \omega^{\alpha}(\mu)$.

Example 3.1 *The system* $\Phi : \mathbf{B}^2 \rightarrow \mathbf{B}^2$ *from Example 1.1 has three omega limit sets,* $\{(0, 1), (0, 0)\}, \{(1, 1)\}, \{(1, 0)\} \in \Omega_{\Phi}$.

Theorem 3.3 *We ask that for arbitrary* $\alpha \in \Pi_n, \beta \in \Pi_n, \mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, k_1 \leq k_2$, *we have* $\forall k \geq k_2$,

$$\phi^{\alpha}(\mu, k) = \phi^{\beta}(\mu', k - k_1).$$

Then $\omega^{\alpha}(\mu) = \omega^{\beta}(\mu')$.

Proof. We fix such arbitrary $\alpha, \beta, \mu, \mu', k_1, k_2$ and we get the existence of $k' \in \mathbf{N}$ with

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k)|k \geq k'\}.$$

Let $k'' \geq \max\{k_2, k'\}$ arbitrary. We infer

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k)|k \geq k''\} = \{\phi^{\beta}(\mu', k - k_1)|k \geq k''\}$$

$$= \{\phi^{\beta}(\mu', k)|k \geq k'' - k_1\} = \omega^{\beta}_{k''-k_1}(\mu'),$$

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k)|k \geq k'' + 1\} = \{\phi^{\beta}(\mu', k - k_1)|k \geq k'' + 1\}$$

$$= \{\phi^{\beta}(\mu', k)|k \geq k'' - k_1 + 1\} = \omega^{\beta}_{k''-k_1+1}(\mu'),$$

...

i.e. $\omega^{\beta}_{k''-k_1}(\mu') = \omega^{\beta}_{k''-k_1+1}(\mu') = ... = \omega^{\beta}(\mu')$ and finally $\omega^{\alpha}(\mu) = \omega^{\beta}(\mu')$. ■

Theorem 3.4 *We suppose that* $A \in \Omega_{\Phi}$ *is terminal and* $\exists \mu \in A, \exists \alpha \in \Pi_n, \omega^{\alpha}(\mu) \wedge A \neq \emptyset$. *Then* $O^{\alpha}(\mu) \vee A$ *is terminal.*

Proof. The hypothesis states the existence of $\beta \in \Pi_n, \mu' \in \mathbf{B}^n$ with $A = \omega^{\beta}(\mu')$, and we suppose that

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k)|k \geq k'\}, \tag{9}$$

$$\omega^{\beta}(\mu') = \{\phi^{\beta}(\mu', k)|k \geq k''\} \tag{10}$$

for suitably chosen $k' \in \mathbf{N}, k'' \in \mathbf{N}$. Let

$$\nu \overset{hyp}{\in} \omega^{\alpha}(\mu) \wedge A \tag{11}$$

arbitrary. Then $k'_1 \geq k'$ exists with

$$\phi^\alpha(\mu, k'_1) \overset{(9),(11)}{=} \nu,$$

$k''_1 \geq 1$ exists such that

$$\alpha^0 \cup ... \cup \alpha^{k''_1 - 1} = (1, ..., 1),$$

and $k'''_1 \in \mathbf{N}$ exists also with

$$O^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \in \{0, ..., k'''_1\}\}.$$

We fix a $k_1 \geq \max\{k'_1, k''_1, k'''_1\}$ from infinitely many such possibilities satisfying

$$\phi^\alpha(\mu, k_1) = \nu, \tag{12}$$

$$\alpha^0 \cup ... \cup \alpha^{k_1 - 1} = (1, ..., 1), \tag{13}$$

$$O^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \in \{0, ..., k_1\}\}. \tag{14}$$

We have also the existence of $k_2 \geq k''$ with

$$\phi^\beta(\mu', k_2) \overset{(10),(11)}{=} \nu \tag{15}$$

and we know that

$$\mu \overset{hyp}{\in} A. \tag{16}$$

Some $k'_3 \geq k_2$ exists that fulfills

$$\omega^\beta(\mu') = \{\phi^\beta(\mu', k) | k \in \{k_2, ..., k'_3\}\},$$

and some $k''_3 \geq k''$ exists with

$$\phi^\beta(\mu', k''_3) \overset{(10),(16)}{=} \mu.$$

We fix $k_3 \geq \max\{k'_3, k''_3\}$ from infinitely many such possibilities, that satisfies

$$\omega^\beta(\mu') = \{\phi^\beta(\mu', k) | k \in \{k_2, ..., k_3\}\}, \tag{17}$$

$$\phi^\beta(\mu', k_3) = \mu. \tag{18}$$

We define $\gamma : \mathbf{N} \to \mathbf{B}^n$ by

$$\forall k \in \{0, ..., k_1 - 1\}, \gamma^k = \alpha^k, \tag{19}$$

$$\forall k \in \{k_1, ..., k_1 - k_2 + k_3 - 1\}, \gamma^k = \beta^{k - k_1 + k_2}, \tag{20}$$

$$...$$

and by the fact that it is periodic, with the period $T = k_1 - k_2 + k_3 : \forall k \in \mathbf{N}$,

$$\gamma^k = \gamma^{k+T}. \tag{21}$$

We have $\gamma \in \Pi_n$, because

$$\gamma^0 \cup ... \cup \gamma^{k_1 - 1} \cup ... \cup \gamma^{k_1 - k_2 + k_3 - 1} \geq \gamma^0 \cup ... \cup \gamma^{k_1 - 1} \overset{(19)}{=} \alpha^0 \cup ... \cup \alpha^{k_1 - 1} \overset{(13)}{=} (1, ..., 1).$$

From $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^\gamma(\mu, k) \overset{(19)}{=} \phi^\alpha(\mu, k), \tag{22}$$

$$\phi^\gamma(\mu, k_1) = \phi^\alpha(\mu, k_1) \overset{(12)}{=} \nu \overset{(15)}{=} \phi^\beta(\mu', k_2), \tag{23}$$

$\forall k \in \{k_1, ..., k_1 - k_2 + k_3 - 1\}$,

$$\phi^{\gamma}(\mu, k) = \phi^{\sigma^{k_1}(\gamma)}(\phi^{\gamma}(\mu, k_1), k - k_1) \overset{(20),(23)}{=} \phi^{\sigma^{k_2}(\beta)}(\phi^{\beta}(\mu', k_2), k - k_1)$$

$$= \phi^{\beta}(\mu', k - k_1 + k_2), \tag{24}$$

$$\phi^{\gamma}(\mu, k_1 - k_2 + k_3) = \phi^{\beta}(\mu', k_3) \overset{(18)}{=} \mu = \phi^{\gamma}(\mu, 0), \tag{25}$$

and from (21) we can prove by induction on $k \in \mathbf{N}$ the periodicity of $\phi^{\gamma}(\mu, \cdot) : \forall k \in \mathbf{N}$,

$$\phi^{\gamma}(\mu, k) = \phi^{\gamma}(\mu, k + T).$$

Moreover:

$$O^{\alpha}(\mu) \vee A \overset{(14),(17)}{=} \{\phi^{\alpha}(\mu, k) | k \in \{0, ..., k_1\}\} \vee \{\phi^{\beta}(\mu', k) | k \in \{k_2, ..., k_3\}\}$$

$$\overset{(22),(23)}{=} \{\phi^{\gamma}(\mu, k) | k \in \{0, ..., k_1\}\} \vee \{\phi^{\beta}(\mu', k - k_1 + k_2) | k \in \{k_1, ..., k_1 - k_2 + k_3\}\}$$

$$\overset{(24),(25)}{=} \{\phi^{\gamma}(\mu, k) | k \in \{0, ..., k_1\}\} \vee \{\phi^{\gamma}(\mu, k) | k \in \{k_1, ..., k_1 - k_2 + k_3\}\}$$

$$= \{\phi^{\gamma}(\mu, k) | k \in \{0, ..., k_1 - k_2 + k_3\}\} = O^{\gamma}(\mu) = \omega^{\gamma}(\mu),$$

and when writing the last equality we have used the periodicity of $\phi^{\gamma}(\mu, \cdot)$. ■

4 Equivalent omega limit sets

Definition 4.1 *We say that the omega limit sets $A \in \Omega_{\Phi}, B \in \Omega_{\Phi}$ are **equivalent**, and we denote this by $A \perp B$, if*

$$\exists \delta \in \Pi_n, \exists \nu \in A, O^{\delta}(\nu) \wedge B \neq \emptyset, \tag{26}$$

$$\exists \delta' \in \Pi_n, \exists \nu' \in B, O^{\delta'}(\nu') \wedge A \neq \emptyset \tag{27}$$

hold.

Remark 4.1 *If the omega limit sets A, B satisfy $A \wedge B \neq \emptyset$, in particular if $A \subset B$, then $A \perp B$. This happens because in (26), (27) we can choose $\nu \in A \wedge B, \nu' \in A \wedge B$ and $\delta \in \Pi_n, \delta' \in \Pi_n$ arbitrary.*

Theorem 4.1 *The relation $\perp \subset \Omega_{\Phi} \times \Omega_{\Phi}$ is an equivalence.*

Proof. The refflexivity and the symmetry of $\perp$ are obvious, we prove transitivity now. We have the existence of $\alpha \in \Pi_n, \beta \in \Pi_n, \gamma \in \Pi_n$, $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n, \mu'' \in \mathbf{B}^n$ and $k' \in \mathbf{N}, k'' \in \mathbf{N}, k''' \in \mathbf{N}$ that satisfy

$$A = \omega^{\alpha}(\mu),$$

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k) | k \geq k'\}, \tag{28}$$

$$B = \omega^{\beta}(\mu'),$$

$$\omega^{\beta}(\mu') = \{\phi^{\beta}(\mu', k) | k \geq k''\}, \tag{29}$$

$$C = \omega^{\gamma}(\mu''),$$

$$\omega^{\gamma}(\mu'') = \{\phi^{\gamma}(\mu'', k) | k \geq k'''\}. \tag{30}$$

The hypothesis states that $A \perp B$ and $B \perp C$ are true:

$$\exists \delta \in \Pi_n, \exists \nu \in \omega^{\alpha}(\mu), O^{\delta}(\nu) \wedge \omega^{\beta}(\mu') \neq \emptyset, \tag{31}$$

$$\exists \xi \in \Pi_n, \exists \lambda \in \omega^{\beta}(\mu'), O^{\xi}(\lambda) \wedge \omega^{\alpha}(\mu) \neq \emptyset, \tag{32}$$

$$\exists\delta' \in \Pi_n, \exists\nu' \in \omega^{\beta}(\mu'), O^{\delta'}(\nu') \wedge \omega^{\gamma}(\mu'') \neq \emptyset, \tag{33}$$

$$\exists\xi' \in \Pi_n, \exists\lambda' \in \omega^{\gamma}(\mu''), O^{\xi'}(\lambda') \wedge \omega^{\beta}(\mu') \neq \emptyset, \tag{34}$$

and we must prove the satisfaction of $A \perp C$:

$$\exists\delta'' \in \Pi_n, \exists\nu'' \in \omega^{\alpha}(\mu), O^{\delta''}(\nu'') \wedge \omega^{\gamma}(\mu'') \neq \emptyset, \tag{35}$$

$$\exists\xi'' \in \Pi_n, \exists\lambda'' \in \omega^{\gamma}(\mu''), O^{\xi''}(\lambda'') \wedge \omega^{\alpha}(\mu) \neq \emptyset. \tag{36}$$

We get the existence of $k_1 \in \mathbf{N}$ and $k_3 > k_2 \geq k''$ such that

$$\phi^{\delta}(\nu, k_1) \overset{(29),(31)}{=} \phi^{\beta}(\mu', k_2), \tag{37}$$

$$\phi^{\beta}(\mu', k_3) \overset{(29),(33)}{=} \nu', \tag{38}$$

and we obtain also the existence of $k_4 \in \mathbf{N}$ and $k_5 \geq k'''$ satisfying

$$\phi^{\delta'}(\nu', k_4) \overset{(30),(33)}{=} \phi^{\gamma}(\mu'', k_5). \tag{39}$$

At this moment we consider the computation function $\delta'' \in \Pi_n$ which is arbitrary and fulfills

$$\delta''^{k} = \begin{cases} \delta^{k}, \ if\ k \in \{0, ..., k_1 - 1\}, \\ \beta^{k-k_1+k_2}, \ if\ k \in \{k_1, ..., k_1 - k_2 + k_3 - 1\}, \\ \delta'^{k-k_1+k_2-k_3}, \ if\ k \in \{k_1 - k_2 + k_3, \\ ..., k_1 - k_2 + k_3 + k_4 - 1\}. \end{cases} \tag{40}$$

We prove the satisfaction of (35) for $\nu'' = \nu$. We infer: $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^{\delta''}(\nu, k) \overset{(40)}{=} \phi^{\delta}(\nu, k),$$

$$\phi^{\delta''}(\nu, k_1) = \phi^{\delta}(\nu, k_1) \overset{(37)}{=} \phi^{\beta}(\mu', k_2), \tag{41}$$

$\forall k \in \{k_1, ..., k_1 - k_2 + k_3 - 1\}$,

$$\phi^{\delta''}(\nu, k) = \phi^{\sigma^{k_1}(\delta'')}(\phi^{\delta''}(\nu, k_1), k - k_1) \overset{(40),(41)}{=} \phi^{\sigma^{k_2}(\beta)}(\phi^{\beta}(\mu', k_2), k - k_1)$$

$$= \phi^{\beta}(\mu', k - k_1 + k_2),$$

$$\phi^{\delta''}(\nu, k_1 - k_2 + k_3) = \phi^{\beta}(\mu', k_3) \overset{(38)}{=} \nu', \tag{42}$$

$\forall k \in \{k_1 - k_2 + k_3, ..., k_1 - k_2 + k_3 + k_4 - 1\}$,

$$\phi^{\delta''}(\nu, k) = \phi^{\sigma^{k_1-k_2+k_3}(\delta'')}(\phi^{\delta''}(\nu, k_1 - k_2 + k_3), k - k_1 + k_2 - k_3)$$

$$\overset{(40),(42)}{=} \phi^{\delta'}(\nu', k - k_1 + k_2 - k_3),$$

$$\phi^{\delta''}(\nu, k_1 - k_2 + k_3 + k_4) = \phi^{\delta'}(\nu', k_4) \overset{(39)}{=} \phi^{\gamma}(\mu'', k_5).$$

As $k_5 \geq k'''$, we have obtained that $O^{\delta''}(\nu) \wedge \omega^{\gamma}(\mu'') \neq \emptyset$. (35) is proved and (36) can be proved similarly. ■

Theorem 4.2 *The omega limit sets $A \in \Omega_{\Phi}, B \in \Omega_{\Phi}$ are given. If $A \perp B$, then the omega limit set $C \in \Omega_{\Phi}$ exists with the property that $A \perp C, B \perp C$ and $A \vee B \subset C$.*

Proof. We have the existence of $\alpha \in \Pi_n, \mu \in \mathbf{B}^n, \beta \in \Pi_n, \mu' \in \mathbf{B}^n$ such that

$$A = \omega^\alpha(\mu),$$

$$B = \omega^\beta(\mu').$$

The hypothesis $A \perp B$ states the existence of $\delta \in \Pi_n, \nu$ such that

$$\nu \in \omega^\alpha(\mu), \tag{43}$$

$$O^\delta(\nu) \wedge \omega^\beta(\mu') \neq \emptyset, \tag{44}$$

and of $\delta' \in \Pi_n, \nu'$ with

$$\nu' \in \omega^\beta(\mu'), \tag{45}$$

$$O^{\delta'}(\nu') \wedge \omega^\alpha(\mu) \neq \emptyset. \tag{46}$$

We want to show the existence of $\gamma \in \Pi_n$ with $C = \omega^\gamma(\mu)$ and $\omega^\alpha(\mu) \vee \omega^\beta(\mu') \subset \omega^\gamma(\mu)$.

We suppose that $k' \in \mathbf{N}$, $k'' \in \mathbf{N}$ satisfy

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \geq k'\}, \tag{47}$$

$$\omega^\beta(\mu') = \{\phi^\beta(\mu', k) | k \geq k''\}. \tag{48}$$

From (43), (47) $k_1 \geq k'$ exists with the property

$$\phi^\alpha(\mu, k_1) = \nu, \tag{49}$$

and from (44), (48) we have the existence of $k_2 \in \mathbf{N}, k_3 \geq k''$ with

$$\phi^\delta(\nu, k_2) = \phi^\beta(\mu', k_3). \tag{50}$$

Statements (45), (48) imply the existence of $k'_4 \geq k''$ such that

$$\phi^\beta(\mu', k'_4) = \nu',$$

and we have also the existence of $k''_4 \geq k_3$ for which

$$\omega^\beta(\mu') = \{\phi^\beta(\mu', k) | k \in \{k_3, ..., k''_4\}\}.$$

These allow us to choose from infinitely many possibilities some $k_4 \geq \max\{k'_4, k''_4\}$ with

$$\phi^\beta(\mu', k_4) = \nu', \tag{51}$$

$$\omega^\beta(\mu') = \{\phi^\beta(\mu', k) | k \in \{k_3, ..., k_4\}\}. \tag{52}$$

Statements (46), (47) give the existence of $k_5 \in \mathbf{N}, k_6 \geq k'$ with

$$\phi^{\delta'}(\nu', k_5) = \phi^\alpha(\mu, k_6). \tag{53}$$

(43), (47) imply that $k'_7 \geq k'$ exists making

$$\phi^\alpha(\mu, k'_7) = \nu$$

true, from the progressiveness of α we get the existence of $k''_7 > k_6$ with

$$\alpha^{k_6} \cup ... \cup \alpha^{k''_7 - 1} = (1, ..., 1),$$

and we know also that $k_7''' \geq k_6$ exists such that

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \in \{k_6, ..., k_7'''\}\}.$$

We fix from infinitely many possibilities some $k_7 \geq \max\{k_7', k_7'', k_7'''\}$ that satisfies

$$\phi^\alpha(\mu, k_7) = \nu \tag{54}$$

$$\alpha^{k_6} \cup ... \cup \alpha^{k_7-1} = (1, ..., 1), \tag{55}$$

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \in \{k_6, ..., k_7\}\}. \tag{56}$$

We define $\gamma : \mathbf{N} \to \mathbf{B}^n$ by

$$\begin{gathered}\forall k \in \{0, ..., k_1 - 1\},\\ \gamma^k = \alpha^k,\end{gathered} \tag{57}$$

$$\begin{gathered}\forall k \in \{k_1, ..., k_1 + k_2 - 1\},\\ \gamma^k = \delta^{k-k_1},\end{gathered} \tag{58}$$

$$\begin{gathered}\forall k \in \{k_1 + k_2, ..., k_1 + k_2 - k_3 + k_4 - 1\},\\ \gamma^k = \beta^{k-k_1-k_2+k_3},\end{gathered} \tag{59}$$

$$\begin{gathered}\forall k \in \{k_1 + k_2 - k_3 + k_4, ..., k_1 + k_2 - k_3 + k_4 + k_5 - 1\},\\ \gamma^k = \delta'^{k-k_1-k_2+k_3-k_4},\end{gathered} \tag{60}$$

$$\begin{gathered}\forall k \in \{k_1 + k_2 - k_3 + k_4 + k_5, ..., k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7 - 1\},\\ \gamma^k = \alpha^{k-k_1-k_2+k_3-k_4-k_5+k_6},\end{gathered} \tag{61}$$

$$...$$

and at this moment the sequence (58),...,(61) repeats with the period

$$T = k_2 - k_3 + k_4 + k_5 - k_6 + k_7. \tag{62}$$

The fact that $\gamma \in \Pi_n$ follows from its eventual periodicity $\forall k \geq k_1$,

$$\gamma^k = \gamma^{k+T} \tag{63}$$

and from

$$\gamma^{k_1} \cup ... \cup \gamma^{k_1+T-1} \overset{(61),(62)}{\geq} \alpha^{k_6} \cup ... \cup \alpha^{k_7-1} \overset{(55)}{=} (1, ..., 1).$$

We infer $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^\gamma(\mu, k) \overset{(57)}{=} \phi^\alpha(\mu, k),$$

$$\phi^\gamma(\mu, k_1) = \phi^\alpha(\mu, k_1) \overset{(49)}{=} \nu, \tag{64}$$

$\forall k \in \{k_1, ..., k_1 + k_2 - 1\}$,

$$\phi^\gamma(\mu, k) = \phi^{\sigma^{k_1}(\gamma)}(\phi^\gamma(\mu, k_1), k - k_1) \overset{(58),(64)}{=} \phi^\delta(\nu, k - k_1),$$

$$\phi^\gamma(\mu, k_1 + k_2) = \phi^\delta(\nu, k_2) \overset{(50)}{=} \phi^\beta(\mu', k_3), \tag{65}$$

$\forall k \in \{k_1 + k_2, ..., k_1 + k_2 - k_3 + k_4 - 1\}$,

$$\phi^\gamma(\mu, k) = \phi^{\sigma^{k_1+k_2}(\gamma)}(\phi^\gamma(\mu, k_1 + k_2), k - k_1 - k_2) \tag{66}$$

$$\overset{(59),(65)}{=} \phi^{\sigma^{k_3}(\beta)}(\phi^\beta(\mu', k_3), k - k_1 - k_2) = \phi^\beta(\mu', k - k_1 - k_2 + k_3),$$

$$\phi^{\gamma}(\mu, k_1 + k_2 - k_3 + k_4) = \phi^{\beta}(\mu', k_4) \overset{(51)}{=} \nu', \tag{67}$$

$\forall k \in \{k_1 + k_2 - k_3 + k_4, ..., k_1 + k_2 - k_3 + k_4 + k_5 - 1\}$,

$$\phi^{\gamma}(\mu, k) = \phi^{\sigma^{k_1+k_2-k_3+k_4}(\gamma)}(\phi^{\gamma}(\mu, k_1 + k_2 - k_3 + k_4), k - k_1 - k_2 + k_3 - k_4)$$

$$\overset{(60),(67)}{=} \phi^{\delta'}(\nu', k - k_1 - k_2 + k_3 - k_4),$$

$$\phi^{\gamma}(\mu, k_1 + k_2 - k_3 + k_4 + k_5) = \phi^{\delta'}(\nu', k_5) \overset{(53)}{=} \phi^{\alpha}(\mu, k_6), \tag{68}$$

$\forall k \in \{k_1 + k_2 - k_3 + k_4 + k_5, ..., k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7 - 1\}$,

$$\phi^{\gamma}(\mu, k) \tag{69}$$

$$= \phi^{\sigma^{k_1+k_2-k_3+k_4+k_5}(\gamma)}(\phi^{\gamma}(\mu, k_1 + k_2 - k_3 + k_4 + k_5), k - k_1 - k_2 + k_3 - k_4 - k_5)$$

$$\overset{(61),(68)}{=} \phi^{\sigma^{k_6}(\alpha)}(\phi^{\alpha}(\mu, k_6), k - k_1 - k_2 + k_3 - k_4 - k_5)$$

$$= \phi^{\alpha}(\mu, k - k_1 - k_2 + k_3 - k_4 - k_5 + k_6),$$

$$\phi^{\gamma}(\mu, k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7) = \phi^{\alpha}(\mu, k_7) \overset{(54)}{=} \nu. \tag{70}$$

We can see that

$$\phi^{\gamma}(\mu, k_1 + (k_2 - k_3 + k_4 + k_5 - k_6 + k_7)) \overset{(62)}{=} \phi^{\gamma}(\mu, k_1 + T)$$

$$\overset{(70)}{=} \nu \overset{(64)}{=} \phi^{\gamma}(\mu, k_1),$$

and the eventual periodicity of $\phi^{\gamma}(\mu, \cdot)$ follows, since we can prove by induction on $k \geq k_1$, taking into account (63), that

$$\phi^{\gamma}(\mu, k) = \phi^{\gamma}(\mu, k + T).$$

We have:

$$\omega^{\beta}(\mu') \overset{(52)}{=} \{\phi^{\beta}(\mu', k) | k \in \{k_3, ..., k_4\}\}$$

$$\overset{(65),(66),(67)}{=} \{\phi^{\gamma}(\mu, k) | k \in \{k_1 + k_2, ..., k_1 + k_2 - k_3 + k_4\}\}$$

$$\subset \{\phi^{\gamma}(\mu, k) | k \in \{k_1, ..., k_1 + T\}\} = \omega^{\gamma}(\mu),$$

$$\omega^{\alpha}(\mu) \overset{(56)}{=} \{\phi^{\alpha}(\mu, k) | k \in \{k_6, ..., k_7\}\}$$

$$\overset{(68),(69),(70)}{=} \{\phi^{\gamma}(\mu, k) | k \in \{k_1 + k_2 - k_3 + k_4 + k_5, ..., k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7\}\}$$

$$\subset \{\phi^{\gamma}(\mu, k) | k \in \{k_1, ..., k_1 + T\}\} = \omega^{\gamma}(\mu),$$

thus the set $C = \omega^{\gamma}(\mu)$ satisfies $A \vee B \subset C$. Remark 4.1 shows that $\omega^{\alpha}(\mu) \perp \omega^{\gamma}(\mu)$ and $\omega^{\beta}(\mu') \perp \omega^{\gamma}(\mu)$ hold. ∎

5 Maximal Omega Limit Sets

Theorem 5.1 *Let $M \in \Omega_\Phi$ an omega limit set. The following statements are equivalent*

$$\forall A \in \Omega_\Phi, A \wedge M \neq \emptyset \Longrightarrow A \subset M, \tag{71}$$

$$\forall A \in \Omega_\Phi, A \perp M \Longrightarrow A \subset M. \tag{72}$$

Proof. We take $A \in \Omega_\Phi$ arbitrary.

(71)$\Longrightarrow$(72) The computation functions $\alpha \in \Pi_n, \beta \in \Pi_n$, the points $\mu \in \mathbf{B}^n, \mu' \in \mathbf{B}^n$ and $k' \in \mathbf{N}, k'' \in \mathbf{N}$ exist satisfying

$$A = \omega^\alpha(\mu),$$

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \geq k'\},$$

$$M = \omega^\beta(\mu'),$$

$$\omega^\beta(\mu') = \{\phi^\beta(\mu', k) | k \geq k''\}.$$

The hypothesis $A \perp M$ means the existence of $\nu \in A : \exists k_1 \geq k'$,

$$\nu = \phi^\alpha(\mu, k_1), \tag{73}$$

and of $\delta \in \Pi_n$ such that $O^\delta(\nu) \wedge M \neq \emptyset : \exists k_2 \in \mathbf{N}, \exists k_3 \geq k''$,

$$\phi^\delta(\nu, k_2) = \phi^\beta(\mu', k_3), \tag{74}$$

and also the existence of $\nu' \in M$, that we can choose without losing the generality to be reached subsequently to $\phi^\beta(\mu', k_3) : \exists k_4 > k_3$,

$$\nu' = \phi^\beta(\mu', k_4), \tag{75}$$

and the existence of $\delta' \in \Pi_n$ such that $O^{\delta'}(\nu') \wedge A \neq \emptyset : \exists k_5 \in \mathbf{N}, \exists k_6 \geq k'$,

$$\phi^{\delta'}(\nu', k_5) = \phi^\alpha(\mu, k_6). \tag{76}$$

We know that $\nu \in A$ is reached again, subsequently to $\phi^\alpha(\mu, k_6) : \exists k_7' > k_6$,

$$\nu = \phi^\alpha(\mu, k_7'),$$

that the progressiveness of α indicates the existence of $k_7'' > k_6$ with

$$\alpha^{k_6} \cup ... \cup \alpha^{k_7''-1} = (1, ..., 1),$$

and also that $k_7''' > k_6$ exists making

$$A = \{\phi^\alpha(\mu, k) | k \in \{k_6, ..., k_7''' - 1\}\}$$

true. We fix from infinitely many possibilities some $k_7 \geq \max\{k_7', k_7'', k_7'''\}$ that satisfies

$$\nu = \phi^\alpha(\mu, k_7), \tag{77}$$

$$\alpha^{k_6} \cup ... \cup \alpha^{k_7-1} = (1, ..., 1), \tag{78}$$

$$A = \{\phi^\alpha(\mu, k) | k \in \{k_6, ..., k_7 - 1\}\}. \tag{79}$$

We define $\gamma : \mathbf{N} \to \mathbf{B}^n$ in the following way:

$$\forall k \in \{0, ..., k_1 - 1\}, \gamma^k = \alpha^k, \tag{80}$$

$$\forall k \in \{k_1, ..., k_1 + k_2 - 1\}, \gamma^k = \delta^{k-k_1}, \tag{81}$$

$$\forall k \in \{k_1 + k_2, ..., k_1 + k_2 - k_3 + k_4 - 1\}, \gamma^k = \beta^{k-k_1-k_2+k_3}, \tag{82}$$

$$\begin{gathered}\forall k \in \{k_1 + k_2 - k_3 + k_4, ..., k_1 + k_2 - k_3 + k_4 + k_5 - 1\},\\ \gamma^k = \delta'^{k-k_1-k_2+k_3-k_4},\end{gathered} \tag{83}$$

$$\begin{gathered}\forall k \in \{k_1 + k_2 - k_3 + k_4 + k_5, ...,\\ k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7 - 1\},\\ \gamma^k = \alpha^{k-k_1-k_2+k_3-k_4-k_5+k_6},\end{gathered} \tag{84}$$

and at this moment the definition of γ is made by repeating (81),...,(84) periodically, with the period

$$T = k_2 - k_3 + k_4 + k_5 - k_6 + k_7 : \tag{85}$$

$\forall k \geq k_1$,

$$\gamma^k = \gamma^{k+T}. \tag{86}$$

We infer that $\gamma \in \Pi_n$ because

$$\gamma^{k_1} \cup ... \cup \gamma^{k_1+T-1} \overset{(84),(85)}{\geq} \alpha^{k_6} \cup ... \cup \alpha^{k_7-1} \overset{(78)}{=} (1, ..., 1).$$

We have: $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^\gamma(\mu, k) \overset{(80)}{=} \phi^\alpha(\mu, k),$$

$$\phi^\gamma(\mu, k_1) = \phi^\alpha(\mu, k_1) \overset{(73)}{=} \nu, \tag{87}$$

$\forall k \in \{k_1, ..., k_1 + k_2 - 1\}$,

$$\phi^\gamma(\mu, k) = \phi^{\sigma^{k_1}(\gamma)}(\phi^\gamma(\mu, k_1), k - k_1) \overset{(81),(87)}{=} \phi^\delta(\nu, k - k_1),$$

$$\phi^\gamma(\mu, k_1 + k_2) = \phi^\delta(\nu, k_2) \overset{(74)}{=} \phi^\beta(\mu', k_3), \tag{88}$$

$\forall k \in \{k_1 + k_2, ..., k_1 + k_2 - k_3 + k_4 - 1\}$,

$$\phi^\gamma(\mu, k) = \phi^{\sigma^{k_1+k_2}(\gamma)}(\phi^\gamma(\mu, k_1 + k_2), k - k_1 - k_2)$$

$$\overset{(82),(88)}{=} \phi^{\sigma^{k_3}(\beta)}(\phi^\beta(\mu', k_3), k - k_1 - k_2) = \phi^\beta(\mu', k - k_1 - k_2 + k_3),$$

$$\phi^\gamma(\mu, k_1 + k_2 - k_3 + k_4) = \phi^\beta(\mu', k_4) \overset{(75)}{=} \nu', \tag{89}$$

$\forall k \in \{k_1 + k_2 - k_3 + k_4, ..., k_1 + k_2 - k_3 + k_4 + k_5 - 1\}$,

$$\phi^\gamma(\mu, k) = \phi^{\sigma^{k_1+k_2-k_3+k_4}(\gamma)}(\phi^\gamma(\mu, k_1 + k_2 - k_3 + k_4), k - k_1 - k_2 + k_3 - k_4)$$

$$\overset{(83),(89)}{=} \phi^{\delta'}(\nu', k - k_1 - k_2 + k_3 - k_4),$$

$$\phi^\gamma(\mu, k_1 + k_2 - k_3 + k_4 + k_5) = \phi^{\delta'}(\nu', k_5) \overset{(76)}{=} \phi^\alpha(\mu, k_6), \tag{90}$$

$\forall k \in \{k_1 + k_2 - k_3 + k_4 + k_5, ..., k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7 - 1\}$,

$$\phi^\gamma(\mu, k)$$

$$= \phi^{\sigma^{k_1+k_2-k_3+k_4+k_5}(\gamma)}(\phi^\gamma(\mu, k_1 + k_2 - k_3 + k_4 + k_5), k - k_1 - k_2 + k_3 - k_4 - k_5)$$

$$\overset{(84),(90)}{=} \phi^{\sigma^{k_6}(\alpha)}(\phi^\alpha(\mu, k_6), k - k_1 - k_2 + k_3 - k_4 - k_5)$$

$$= \phi^\alpha(\mu, k - k_1 - k_2 + k_3 - k_4 - k_5 + k_6),$$

$$\phi^{\gamma}(\mu, k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7) = \phi^{\alpha}(\mu, k_7) \overset{(77)}{=} \nu. \tag{91}$$

We note that

$$\phi^{\gamma}(\mu, k_1 + (k_2 - k_3 + k_4 + k_5 - k_6 + k_7)) \overset{(85)}{=} \phi^{\gamma}(\mu, k_1 + T) \overset{(91)}{=} \nu \overset{(87)}{=} \phi^{\gamma}(\mu, k_1)$$

and we can prove by induction on $k \geq k_1$, by taking into account (86), that

$$\phi^{\gamma}(\mu, k) = \phi^{\gamma}(\mu, k + T).$$

We have constructed an omega limit set

$$C = \omega^{\gamma}(\mu),$$

$$\omega^{\gamma}(\mu) = \{\phi^{\gamma}(\mu, k) | k \geq k_1\}$$

that satisfies $C \wedge M \neq \emptyset$, from

$$\omega^{\gamma}(\mu) \ni \phi^{\gamma}(\mu, k_1 + k_2) \overset{(88)}{=} \phi^{\delta}(\nu, k_2) \overset{(74)}{=} \phi^{\beta}(\mu', k_3) \in \omega^{\beta}(\mu')$$

for example. The hypothesis (71) implies $C \subset M$. But $A \subset C$, since

$$A \overset{(79)}{=} \{\phi^{\alpha}(\mu, k) | k \in \{k_6, ..., k_7 - 1\}\}$$

$$= \{\phi^{\alpha}(\mu, k - k_1 - k_2 + k_3 - k_4 - k_5 + k_6) | k \in \{k_1 + k_2 - k_3 + k_4 + k_5,$$

$$..., k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7 - 1\}\}$$

$$= \{\phi^{\gamma}(\mu, k) | k \in \{k_1 + k_2 - k_3 + k_4 + k_5, ..., k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7 - 1\}\}$$

$$\subset \{\phi^{\gamma}(\mu, k) | k \in \{k_1, ..., k_1 + k_2 - k_3 + k_4 + k_5 - k_6 + k_7 - 1\}\}$$

$$= \{\phi^{\gamma}(\mu, k) | k \in \{k_1, ..., k_1 + T - 1\}\} = \omega^{\gamma}(\mu) = C,$$

thus $A \subset M$.

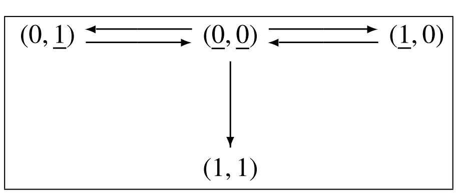
(72)$\Longrightarrow$(71) The hypothesis states that $A \wedge M \neq \emptyset$ and in this case $A \perp M$ holds. The implication (72) shows that $A \subset M$. ■

Definition 5.1 *If one of (71), (72) is true, the set $M \in \Omega_{\Phi}$ is called **maximal**.*

Notation 5.1 *The set of the maximal omega limit sets of Φ is denoted by Ω^{Φ} :*

$$\Omega^{\Phi} = \{M | M \in \Omega_{\Phi}, \forall A \in \Omega_{\Phi}, A \perp M \Longrightarrow A \subset M\}.$$

Example 5.1 *In the case of the system*



the sets $A = \{(0,0), (0,1)\}$, $B = \{(0,0), (1,0)\}$ are omega limit, not maximal, equivalent, and the sets $M_1 = \{(0,1), (0,0), (1,0)\}$, $M_2 = \{(1,1)\}$ are omega limit maximal.

Theorem 5.2 *The maximal omega limit sets are disjoint two by two.*

Proof. In case that $M \in \Omega^{\Phi}$ and $M' \in \Omega^{\Phi}$ arbitrary fulfill $M \wedge M' \neq \emptyset$, the inclusions $M \subset M'$, $M' \subset M$ are both true, therefore $M = M'$. ■

Theorem 5.3 $\forall A \in \Omega_{\Phi}, \forall M \in \Omega^{\Phi}$ *the statements*

$$A \wedge M \neq \emptyset, \tag{92}$$

$$A \perp M, \tag{93}$$

$$A \subset M \tag{94}$$

are equivalent.

Proof. Indeed, for any $A \in \Omega_{\Phi}$ and $M \in \Omega^{\Phi}$, the implications (92)$\Longrightarrow$ (93)$\Longrightarrow$(94)$\Longrightarrow$(92) are obvious. ■

Theorem 5.4 *If* $M \in \Omega^{\Phi}$ *is a maximal omega limit set and* $\exists \mu \in M, \exists \alpha \in \Pi_n, O^{\alpha}(\mu) \setminus M \neq \emptyset$, *then* $\omega^{\alpha}(\mu) \wedge M = \emptyset$.

Proof. The maximal omega limit set M fulfills that $\mu \in M$ and $\alpha \in \Pi_n$ exist such that $O^{\alpha}(\mu) \setminus M \neq \emptyset$. If, against all reason, $\omega^{\alpha}(\mu) \wedge M \neq \emptyset$, then we infer, by applying Theorem 3.4, that $M \vee O^{\alpha}(\mu)$ is omega limit and $M \subsetneq M \vee O^{\alpha}(\mu)$. The last assertion represents a contradiction with the maximality of M. $\omega^{\alpha}(\mu) \wedge M = \emptyset$ follows. ■

6 Omega limit sets vs maximal omega limit sets

Theorem 6.1 *(a) For any* $A \in \Omega_{\Phi}$, *at least one* $M \in \Omega^{\Phi}$ *exists with* $A \wedge M \neq \emptyset$.
(b) For any $A \in \Omega_{\Phi}$, *at most one* $M \in \Omega^{\Phi}$ *exists such that* $A \wedge M \neq \emptyset$.

Proof. We fix an arbitrary $A \in \Omega_{\Phi}$.

(a) If A is maximal, then item (a) is proved, thus we can suppose that $A \notin \Omega^{\Phi}$. In case that $\forall B \in \Omega_{\Phi} \setminus \{C | C \in \Omega_{\Phi}, C \subset A\}, A$ and B are not equivalent, we get that A is maximal, contradiction, therefore some $B \in \Omega_{\Phi} \setminus \{C | C \in \Omega_{\Phi}, C \subset A\}$ exists with $A \perp B$. Theorem 4.2 shows the existence of $A' \in \Omega_{\Phi}$ that fulfills $A \perp A', B \perp A'$ and $A \subsetneq A \vee B \subset A'$.

If A' is maximal, then item (a) is proved, as far as $A \wedge A' \neq \emptyset$, thus we can suppose that $A' \notin \Omega^{\Phi}$. If $\forall B' \in \Omega_{\Phi} \setminus \{C | C \in \Omega_{\Phi}, C \subset A'\}$, $A' \perp B'$ is false, then $A' \in \Omega^{\Phi}$, contradiction, thus $B' \in \Omega_{\Phi} \setminus \{C | C \in \Omega_{\Phi}, C \subset A'\}$ exists with $A' \perp B'$. We infer from Theorem 4.2 the existence of $A'' \in \Omega_{\Phi}$ that fulfills $A' \perp A'', B' \perp A''$ and $A' \subsetneq A' \vee B' \subset A''$.

If A'' is maximal, then (a) is proved because $A \wedge A'' \neq \emptyset$, thus we can suppose that A'' is not maximal...

In finitely many steps we obtain the existence of $A''' \in \Omega^{\Phi}$ that satisfies $A \subsetneq A' \subsetneq A'' \subsetneq \ldots \subsetneq A'''$ and, since in this case $A \wedge A''' \neq \emptyset$, (a) is proved.

(b) Let the arbitrary sets $M \in \Omega^{\Phi}, M' \in \Omega^{\Phi}$ that fulfill $A \wedge M \neq \emptyset, A \wedge M' \neq \emptyset$. From the definition (71) of the maximal omega limit sets we infer that $A \subset M, A \subset M'$ are true, in other words $M \wedge M' \neq \emptyset$ holds. But in this situation $M \subset M'$ and $M' \subset M$ are both true, i.e. $M = M'$. ■

Corollary 6.1 *For any omega limit set* $A \in \Omega_{\Phi}$, *exactly one* $M \in \Omega^{\Phi}$ *exists such that the statements*

$$A \wedge M \neq \emptyset,$$

$$A \perp M,$$

$$A \subset M$$

are true.

Proof. Theorem 6.1 shows that for any $A \in \Omega_{\Phi}$, exactly one $M \in \Omega^{\Phi}$ exists with $A \wedge M \neq \emptyset$. We use Theorem 5.3. ■

7 The set of the omega limit sets of a point

Definition 7.1 *We denote for any $\mu \in \mathbf{B}^n$:*

$$\omega^+(\mu) = \bigvee_{\alpha\in\Pi_n} \omega^\alpha(\mu).$$

$\omega^+(\mu)$ is called the ***set of the omega limit sets of (the point)*** *μ.*

Theorem 7.1 *Let $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ and $\mu \in \mathbf{B}^n$. If $\gamma \in \Pi_n$ and $\mu' \in \mathbf{B}^n$ exist such that $\omega^+(\mu) = \omega^\gamma(\mu')$, then $\omega^+(\mu)$ is invariant.*

Proof. Let $\lambda \in \mathbf{B}^n$ and $\nu \in \omega^+(\mu)$ arbitrary, thus $\alpha \in \Pi_n$ exists with $\nu \in \omega^\alpha(\mu)$. For

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k)|k \geq k'\},$$

where $k' \in \mathbf{N}$ is suitably chosen, we note the existence of $k_1 > k'$ having the property that

$$\phi^\alpha(\mu, k_1) = \nu. \tag{95}$$

We consider now $\beta \in \Pi_n$ arbitrary, satisfying

$$\forall k \in \{0, ..., k_1 - 1\}, \beta^k = \alpha^k,$$

$$\beta^{k_1} = \lambda,$$

i.e. $\nu, \Phi^\lambda(\nu) \in O^\beta(\mu)$, and in addition

$$\phi^\beta(\mu, k_1) = \phi^\alpha(\mu, k_1) = \nu. \tag{96}$$

For

$$\omega^\beta(\mu) = \{\phi^\beta(\mu, k)|k \geq k''\},$$

$k'' \in \mathbf{N}$, the time instant $k_2 > \max\{k_1, k''\}$ exists with $\phi^\beta(\mu, k_2) \in \omega^\beta(\mu)$.

At this moment we take profit of the fact that $\omega^\alpha(\mu), \omega^\beta(\mu) \subset \omega^+(\mu) = \omega^\gamma(\mu')$ is true, from the hypothesis, where

$$\omega^\gamma(\mu') = \{\phi^\gamma(\mu', k)|k \geq k'''\},$$

for some $k''' \in \mathbf{N}$. This means the existence of $k_3 \geq k'''$, $k_4' \geq k'''$, $k_4'' > k_3$ such that

$$\phi^\beta(\mu, k_2) = \phi^\gamma(\mu', k_3), \tag{97}$$

$$\phi^\gamma(\mu', k_4') = \nu,$$

$$\gamma^{k_3} \cup ... \cup \gamma^{k_4''-1} = (1, ..., 1)$$

and we choose from infinitely many possibilities a $k_4 \geq \max\{k_4', k_4''\}$ making

$$\phi^\gamma(\mu', k_4) = \nu, \tag{98}$$

$$\gamma^{k_3} \cup ... \cup \gamma^{k_4-1} = (1, ..., 1) \tag{99}$$

true. We define the computation function $\delta : \mathbf{N} \to \mathbf{B}^n$ in the following manner:

$$\forall k \in \{0, ..., k_1 - 1\}, \delta^k = \alpha^k, \tag{100}$$

$$\forall k \in \{k_1, ..., k_2 - 1\}, \delta^k = \beta^k, \tag{101}$$

$$\forall k \in \{k_2, ..., k_2 - k_3 + k_4 - 1\}, \delta^k = \gamma^{k-k_2+k_3}, \tag{102}$$

and $\forall k \geq k_1$,

$$\delta^k = \delta^{k+T}, \tag{103}$$

where

$$T = -k_1 + k_2 - k_3 + k_4. \tag{104}$$

The fact that $\delta \in \Pi_n$ results from

$$\delta^{k_1} \cup ... \cup \delta^{k_2-k_3+k_4-1} \geq \delta^{k_2} \cup ... \cup \delta^{k_2-k_3+k_4-1} \overset{(102)}{=} \gamma^{k_3} \cup ... \cup \gamma^{k_4-1} \overset{(99)}{=} (1, ..., 1).$$

The values of the state function $\phi^{\delta}(\mu, \cdot)$ are the following: $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^{\delta}(\mu, k) \overset{(100)}{=} \phi^{\alpha}(\mu, k),$$

$$\phi^{\delta}(\mu, k_1) = \phi^{\alpha}(\mu, k_1) \overset{(95)}{=} \nu, \tag{105}$$

$\forall k \in \{k_1, ..., k_2 - 1\}$,

$$\phi^{\delta}(\mu, k) = \phi^{\sigma^{k_1}(\delta)}(\phi^{\delta}(\mu, k_1), k - k_1) \overset{(101),(105),(96)}{=} \phi^{\sigma^{k_1}(\beta)}(\phi^{\beta}(\mu, k_1), k - k_1)$$

$$= \phi^{\beta}(\mu, k),$$

$$\phi^{\delta}(\mu, k_2) = \phi^{\beta}(\mu, k_2) \overset{(97)}{=} \phi^{\gamma}(\mu', k_3), \tag{106}$$

$\forall k \in \{k_2, ..., k_2 - k_3 + k_4 - 1\}$,

$$\phi^{\delta}(\mu, k) = \phi^{\sigma^{k_2}(\delta)}(\phi^{\delta}(\mu, k_2), k - k_2) \overset{(102),(106)}{=} \phi^{\sigma^{k_3}(\gamma)}(\phi^{\gamma}(\mu', k_3), k - k_2)$$

$$= \phi^{\gamma}(\mu', k - k_2 + k_3),$$

$$\phi^{\delta}(\mu, k_2 - k_3 + k_4) = \phi^{\gamma}(\mu', k_4) \overset{(98)}{=} \nu. \tag{107}$$

We see that

$$\phi^{\delta}(\mu, k_1) \overset{(105)}{=} \nu \overset{(107)}{=} \phi^{\delta}(\mu, k_2 - k_3 + k_4)$$

$$= \phi^{\delta}(\mu, k_1 + (-k_1 + k_2 - k_3 + k_4)) \overset{(104)}{=} \phi^{\delta}(\mu, k_1 + T)$$

and we can prove by induction on $k \geq k_1$, by using (103), that

$$\phi^{\delta}(\mu, k) = \phi^{\delta}(\mu, k + T).$$

The conclusion is $\forall p \in \mathbf{N}$,

$$\nu = \phi^{\delta}(\mu, k_1 + pT),$$

$$\Phi^{\lambda}(\nu) = \phi^{\delta}(\mu, k_1 + 1 + pT),$$

i.e.

$$\nu, \Phi^{\lambda}(\nu) \in \{\phi^{\delta}(\mu, k) | k \geq k_1\} = \omega^{\delta}(\mu) \subset \omega^{+}(\mu),$$

therefore the invariance of $\omega^{+}(\mu)$ follows. ■

Theorem 7.2 *Let $\Omega^{\Phi} = \{M_1, ..., M_p\}$ be the set of the maximal omega limit sets of the system Φ. Then $\forall \mu \in \mathbf{B}^n$, the indexes $i_1, i_2, ..., i_q \in \{1, ..., p\}$ exist such that*

$$\omega^{+}(\mu) = M_{i_1} \vee M_{i_2} \vee ... \vee M_{i_q}.$$

Proof. We fix an arbitrary $\mu \in \mathbf{B}^n$.

For any $\alpha \in \Pi_n$, exactly one $i \in \{1, ..., p\}$ exists with

$$\omega^\alpha(\mu) \subset M_i,$$

from Corollary 6.1. This gives the possibility of defining the set $\{i_1, ..., i_q\}$ in the following way:

$$\{i_1, ..., i_q\} = \{i | i \in \{1, ..., p\}, \exists \alpha \in \Pi_n, \omega^\alpha(\mu) \subset M_i\}. \tag{108}$$

We fix now $i \in \{i_1, ..., i_q\}$, arbitrary. M_i is terminal

$$\exists \gamma \in \Pi_n, \exists \mu' \in \mathbf{B}^n, M_i = \omega^\gamma(\mu'),$$

and $\alpha \in \Pi_n$ exists, from (108), such that $\omega^\alpha(\mu) \subset M_i$. We suppose that

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \geq k'\},$$

$$\omega^\gamma(\mu') = \{\phi^\gamma(\mu', k) | k \geq k''\},$$

with $k' \in \mathbf{N}, k'' \in \mathbf{N}$ suitably chosen, and we have the existence of $k_1 \geq k', k_2 \geq k''$ with

$$\phi^\alpha(\mu, k_1) = \phi^\gamma(\mu', k_2). \tag{109}$$

We define $\beta \in \Pi_n$ like this:

$$\beta^k = \begin{cases} \alpha^k, \ if \ k \in \{0, ..., k_1 - 1\}, \\ \gamma^{k-k_1+k_2}, \ if \ k \geq k_1 \end{cases} \tag{110}$$

and we deduce in succession $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^\beta(\mu, k) \overset{(110)}{=} \phi^\alpha(\mu, k),$$

$$\phi^\beta(\mu, k_1) = \phi^\alpha(\mu, k_1) \overset{(109)}{=} \phi^\gamma(\mu', k_2), \tag{111}$$

$\forall k \geq k_1$,

$$\begin{aligned} \phi^\beta(\mu, k) &= \phi^{\sigma^{k_1}(\beta)}(\phi^\beta(\mu, k_1), k - k_1) \\ &\overset{(110),(111)}{=} \phi^{\sigma^{k_2}(\gamma)}(\phi^\gamma(\mu', k_2), k - k_1) = \phi^\gamma(\mu', k - k_1 + k_2). \end{aligned} \tag{112}$$

We have obtained the existence of β, namely the one defined by (110), satisfying

$$M_i = \omega^\gamma(\mu') \overset{Theorem\ 3.3,(112)}{=} \omega^\beta(\mu).$$

Let us denote with $\Pi_n^{i_1}, ..., \Pi_n^{i_q}$ the partition of Π_n defined by $\forall j \in \{i_1, ..., i_q\}$,

$$\Pi_n^j = \{\delta | \delta \in \Pi_n, \omega^\delta(\mu) \subset M_j\},$$

therefore

$$\forall \delta \in \Pi_n^j, \omega^\delta(\mu) \subset M_j. \tag{113}$$

The conclusion is that $\beta' \in \Pi_n^{i_1}, ..., \beta'' \in \Pi_n^{i_q}$ exist such that $M_{i_1} = \omega^{\beta'}(\mu), ..., M_{i_q} = \omega^{\beta''}(\mu)$ and

$$M_{i_1} \vee ... \vee M_{i_q} = \omega^{\beta'}(\mu) \vee ... \vee \omega^{\beta''}(\mu) \subset \omega^+(\mu) = \bigvee_{\delta \in \Pi_n^{i_1} \vee ... \vee \Pi_n^{i_q}} \omega^\delta(\mu)$$

$$= \bigvee_{\delta \in \Pi_n^{i_1}} \omega^\delta(\mu) \vee ... \vee \bigvee_{\delta \in \Pi_n^{i_q}} \omega^\delta(\mu) \overset{(113)}{\subset} M_{i_1} \vee ... \vee M_{i_q}.$$

■

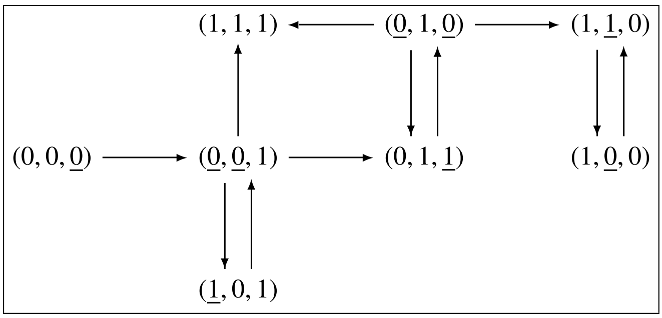
8 Final sets

Definition 8.1 *If the omega limit set $A \in \Omega_\Phi$ is invariant, then it is said to be* ***final,*** *otherwise A is said to be* ***pseudo-final****.*

Notation 8.1 *We denote with F^Φ the set of the final sets of Φ :*

$$F^\Phi = \{A | A \in \Omega_\Phi \text{ and } \forall \lambda \in \mathbf{B}^n, \Phi^\lambda(A) \subset A\}.$$

Example 8.1 *The system*



has four maximal omega limit sets: $M_1 = \{(0,0,1),(1,0,1)\}$, $M_2 = \{(1,1,1)\}$, $M_3 = \{(0,1,1),(0,1,0)\}$, $M_4 = \{(1,1,0),(1,0,0)\}$ two of which, M_2 and M_4, are final. Note the way that, starting from the initial value $\mu = (0,0,0)$, a state $\phi^\alpha(\mu,\cdot)$ meets $0,1$ or 2 pseudo-final sets and at most a final set.

9 The existence of the final sets

Theorem 9.1 *Let $\Omega^\Phi = \{M_1, ..., M_p\}$ the set of the maximal omega limit sets of Φ. Then at least one of them is final.*

Proof. If M_1 is final, the theorem is proved, thus we suppose that it is not, and $\alpha \in \Pi_n, \mu \in M_1$ exist with the property that $O^\alpha(\mu) \setminus M_1 \neq \emptyset$. At this moment Theorem 5.4 states that $\omega^\alpha(\mu) \wedge M_1 = \emptyset$. We suppose without losing the generality, from Corollary 6.1, that $\omega^\alpha(\mu) \subset M_2$.

If M_2 is final, the conclusion of the theorem follows, therefore we can suppose that M_2 is not invariant. In this situation $\alpha' \in \Pi_n, \mu' \in M_2$ exist, having the property $O^{\alpha'}(\mu') \setminus M_2 \neq \emptyset$ thus, from Theorem 5.4, $\omega^{\alpha'}(\mu') \wedge M_2 = \emptyset$. The inclusion $\omega^{\alpha'}(\mu') \subset M_1$, which might be a consequence of Corollary 6.1, produces the situation $M_1 \perp M_2$ which is in contradiction with the maximality of M_1, M_2. We conclude that the only possibility is, without losing the generality, $\omega^{\alpha'}(\mu') \subset M_3$.

...

If M_p is final, the theorem is proved, thus we suppose that this is not the case. Then $\alpha'' \in \Pi_n, \mu'' \in M_p$ exist with $O^{\alpha''}(\mu'') \setminus M_p \neq \emptyset$, i.e. $\omega^{\alpha''}(\mu'') \wedge M_p = \emptyset$. We have been brought to the conclusion, due to Corollary 6.1, that $\omega^{\alpha''}(\mu'')$ is included in one of $M_1, ..., M_{p-1}$ representing, via the equivalence $\perp$, a contradiction with the maximality of these sets.

We have obtained that at least one of $M_1, ..., M_p$ is final. ■

10 The final sets are maximal

Theorem 10.1 *If the set A is final, then it is maximal (as omega limit set): $F^\Phi \subset \Omega^\Phi$.*

Proof. As A is terminal, $\alpha \in \Pi_n$ and $\mu \in \mathbf{B}^n$ exist with $A = \omega^\alpha(\mu)$. We consider an arbitrary terminal set $B \subset \mathbf{B}^n$, for which the truth of

$$B \perp A \Longrightarrow B \subset A$$

must be proved. In this respect let $\beta \in \Pi_n, \mu' \in \mathbf{B}^n$ arbitrary, with $B = \omega^\beta(\mu')$. We have

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \geq k'\},$$

$$\omega^\beta(\mu') = \{\phi^\beta(\mu', k) | k \geq k''\}, \tag{114}$$

with $k' \in \mathbf{N}, k'' \in \mathbf{N}$ suitably chosen. The hypothesis states

$$\exists \delta \in \Pi_n, \exists \nu \in \omega^\beta(\mu'), O^\delta(\nu) \wedge \omega^\alpha(\mu) \neq \emptyset,$$

$$\exists \delta' \in \Pi_n, \exists \nu' \in \omega^\alpha(\mu), O^{\delta'}(\nu') \wedge \omega^\beta(\mu') \neq \emptyset, \tag{115}$$

and we suppose against all reason that $B \subset A$ is false: $\omega^\beta(\mu') \setminus \omega^\alpha(\mu) \neq \emptyset$, i.e. $\nu'' \in \omega^\beta(\mu') \setminus \omega^\alpha(\mu)$ exists.

From (114), (115) we get the existence of $\nu' \in \omega^\alpha(\mu)$, $k_1 \in \mathbf{N}$ and $k_3 > k_2 \geq k''$ such that

$$\phi^{\delta'}(\nu', k_1) = \phi^\beta(\mu', k_2), \tag{116}$$

$$\phi^\beta(\mu', k_3) = \nu''. \tag{117}$$

We take an arbitrary $\gamma \in \Pi_n$ now, that satisfies

$$\gamma^k = \begin{cases} \delta'^k, \; if \; k \in \{0, ..., k_1 - 1\}, \\ \beta^{k-k_1+k_2}, \; if \; k \in \{k_1, ..., k_1 - k_2 + k_3 - 1\}. \end{cases} \tag{118}$$

We infer: $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^\gamma(\nu', k) \stackrel{(118)}{=} \phi^{\delta'}(\nu', k),$$

$$\phi^\gamma(\nu', k_1) = \phi^{\delta'}(\nu', k_1) \stackrel{(116)}{=} \phi^\beta(\mu', k_2), \tag{119}$$

$\forall k \in \{k_1, ..., k_1 - k_2 + k_3 - 1\}$,

$$\phi^\gamma(\nu', k) = \phi^{\sigma^{k_1}(\gamma)}(\phi^\gamma(\mu', k_1), k - k_1) \stackrel{(118),(119)}{=} \phi^{\sigma^{k_2}(\beta)}(\phi^\beta(\mu', k_2), k - k_1)$$

$$= \phi^\beta(\mu', k - k_1 + k_2),$$

$$\phi^\gamma(\nu', k_1 - k_2 + k_3) = \phi^\beta(\mu', k_3) \stackrel{(117)}{=} \nu''.$$

The last equation is a contradiction with the invariance of A, in the sense that $k \in \{0, ..., k_1 - k_2 + k_3 - 1\}$ exists such that $\phi^\gamma(\nu', k) \in A, \phi^\gamma(\nu', k+1) = \Phi^{\gamma^k}(\phi^\gamma(\nu', k)) \notin A$. We have obtained that $\omega^\beta(\mu') \setminus \omega^\alpha(\mu) = \emptyset$, thus $B \subset A$ holds. ■

11 The set of the omega limit sets of a point revisited

Theorem 11.1 *Let $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, $\Omega^\Phi = \{M_1, ..., M_p\}$, and $\mu \in \mathbf{B}^n$. In the equation*

$$\omega^+(\mu) = M_{i_1} \vee M_{i_2} \vee ... \vee M_{i_q}$$

which is known to be true from Theorem 7.2, where $i_1, i_2, ..., i_q \in \{1, ..., p\}$, at least one of $M_{i_1}, M_{i_2}, ..., M_{i_q}$ is final.

Proof. We fix $\alpha \in \Pi_n$ arbitrary and we suppose, without losing the generality, that

$$\omega^\alpha(\mu) \subset M_{i_1}. \tag{120}$$

We have the existence of $l' \in \mathbf{N}$ with

$$\omega^\alpha(\mu) = \{\phi^\alpha(\mu, k) | k \geq l'\}.$$

If $\forall\alpha' \in \Pi_n, \forall\mu' \in M_{i_1}, O^{\alpha'}(\mu') \subset M_{i_1}$, then M_{i_1} is final and the theorem is proved, thus we can suppose that this is false and $\alpha' \in \Pi_n, \mu' \in M_{i_1}$ exist with $O^{\alpha'}(\mu') \smallsetminus M_{i_1} \neq \emptyset$. In this situation Theorem 5.4 states that $\omega^{\alpha'}(\mu') \wedge M_{i_1} = \emptyset$. The request that M_{i_1} is omega limit set means the existence of $\beta' \in \Pi_n, \nu' \in \mathbf{B}^n$ and $k' \in \mathbf{N}$ with

$$M_{i_1} = \omega^{\beta'}(\nu'),$$

$$\omega^{\beta'}(\nu') = \{\phi^{\beta'}(\nu', k) | k \geq k'\}.$$

Then $l_1 \geq l'$ and $k_2 > k_1 \geq k'$ exist such that

$$\phi^{\alpha}(\mu, l_1) \overset{(120)}{=} \phi^{\beta'}(\nu', k_1), \tag{121}$$

$$\phi^{\beta'}(\nu', k_2) \overset{(120)}{=} \mu', \tag{122}$$

and we define

$$\gamma^k = \begin{cases} \alpha^k, \ if\ k \in \{0, ..., l_1 - 1\}, \\ \beta'^{k-l_1+k_1}, \ if\ k \in \{l_1, ..., l_1 - k_1 + k_2 - 1\}, \\ \alpha'^{k-l_1+k_1-k_2}, \ if\ k \geq l_1 - k_1 + k_2 \end{cases} \tag{123}$$

for which we obtain $\forall k \in \{0, ..., l_1 - 1\}$,

$$\phi^{\gamma}(\mu, k) \overset{(123)}{=} \phi^{\alpha}(\mu, k), \tag{124}$$

$$\phi^{\gamma}(\mu, l_1) = \phi^{\alpha}(\mu, l_1) \overset{(121)}{=} \phi^{\beta'}(\nu', k_1), \tag{125}$$

$\forall k \in \{l_1, ..., l_1 - k_1 + k_2 - 1\}$,

$$\phi^{\gamma}(\mu, k) = \phi^{\sigma^{l_1}(\gamma)}(\phi^{\gamma}(\mu, l_1), k - l_1) \tag{126}$$

$$\overset{(123),(125)}{=} \phi^{\sigma^{k_1}(\beta')}(\phi^{\beta'}(\nu', k_1), k - l_1) = \phi^{\beta'}(\nu', k - l_1 + k_1),$$

$$\phi^{\gamma}(\mu, l_1 - k_1 + k_2) = \phi^{\beta'}(\nu', k_2) \overset{(122)}{=} \mu', \tag{127}$$

$\forall k \geq l_1 - k_1 + k_2$,

$$\begin{aligned} \phi^{\gamma}(\mu, k) &= \phi^{\sigma^{l_1-k_1+k_2}(\gamma)}(\phi^{\gamma}(\mu, l_1 - k_1 + k_2), k - l_1 + k_1 - k_2) \\ &\overset{(123),(127)}{=} \phi^{\alpha'}(\mu', k - l_1 + k_1 - k_2). \end{aligned} \tag{128}$$

We infer

$$\omega^{\gamma}(\mu) \overset{Theorem\ 3.3,(128)}{=} \omega^{\alpha'}(\mu').$$

We suppose now without losing the generality that

$$\omega^{\alpha'}(\mu') \subset M_{i_2}. \tag{129}$$

Then $l'' \in \mathbf{N}$ exists with

$$\omega^{\alpha'}(\mu') = \{\phi^{\alpha'}(\mu', k) | k \geq l''\}.$$

If $\forall\alpha'' \in \Pi_n, \forall\mu'' \in M_{i_2}, O^{\alpha''}(\mu'') \subset M_{i_2}$, then M_{i_2} is final and the theorem is proved, thus we can suppose that $\alpha'' \in \Pi_n, \mu'' \in M_{i_2}$ exist with $O^{\alpha''}(\mu'') \smallsetminus M_{i_2} \neq \emptyset$ therefore, from Theorem 5.4, $\omega^{\alpha''}(\mu'') \wedge M_{i_2} = \emptyset$. The fact that M_{i_2} is terminal asks the existence of $\beta'' \in \Pi_n, \nu'' \in \mathbf{B}^n$ and $k'' \in \mathbf{N}$ with

$$M_{i_2} = \omega^{\beta''}(\nu''),$$

$$\omega^{\beta''}(\nu'') = \{\phi^{\beta''}(\nu'', k) | k \geq k''\}.$$

We have the existence of $l'_1 \geq l''$ and $k'_2 > k'_1 \geq k''$ satisfying

$$\phi^{\alpha'}(\mu', l'_1) \overset{(129)}{=} \phi^{\beta''}(\nu'', k'_1), \tag{130}$$

$$\phi^{\beta''}(\nu'', k'_2) = \mu'' \tag{131}$$

and we define γ' in the following manner:

$$\gamma'^k = \begin{cases} \gamma^k, \ if \ k \in \{0, ..., l_1 + l'_1 - k_1 + k_2 - 1\}, \\ \beta''^{k-l_1-l'_1+k_1+k'_1-k_2}, \ if \ k \in \{l_1 + l'_1 - k_1 + k_2, ..., \\ l_1 + l'_1 - k_1 - k'_1 + k_2 + k'_2 - 1\}, \\ \alpha''^{k-l_1-l'_1+k_1+k'_1-k_2-k'_2}, \\ if \ k \geq l_1 + l'_1 - k_1 - k'_1 + k_2 + k'_2. \end{cases} \tag{132}$$

We get $\forall k \in \{0, ..., l_1 + l'_1 - k_1 + k_2 - 1\}$,

$$\begin{aligned} \phi^{\gamma'}(\mu, k) &\overset{(132)}{=} \phi^{\gamma}(\mu, k), \\ \phi^{\gamma'}(\mu, l_1 + l'_1 - k_1 + k_2) &= \phi^{\gamma}(\mu, l_1 + l'_1 - k_1 + k_2) \\ &\overset{(128)}{=} \phi^{\alpha'}(\mu', l'_1) \overset{(130)}{=} \phi^{\beta''}(\nu'', k'_1), \end{aligned} \tag{133}$$

$\forall k \in \{l_1 + l'_1 - k_1 + k_2, ..., l_1 + l'_1 - k_1 - k'_1 + k_2 + k'_2 - 1\}$,

$$\begin{aligned} \phi^{\gamma'}(\mu, k) &= \phi^{\sigma^{l_1+l'_1-k_1+k_2}(\gamma')}(\phi^{\gamma'}(\mu, l_1 + l'_1 - k_1 + k_2), k - l_1 - l'_1 + k_1 - k_2) \\ &\overset{(132),(133)}{=} \phi^{\sigma^{k'_1}(\beta'')}(\phi^{\beta''}(\nu'', k'_1), k - l_1 - l'_1 + k_1 - k_2) \\ &= \phi^{\beta''}(\nu'', k - l_1 - l'_1 + k_1 + k'_1 - k_2), \end{aligned}$$

$$\phi^{\gamma'}(\mu, l_1 + l'_1 - k_1 - k'_1 + k_2 + k'_2) = \phi^{\beta''}(\nu'', k'_2) \overset{(131)}{=} \mu'', \tag{134}$$

$\forall k \geq l_1 + l'_1 - k_1 - k'_1 + k_2 + k'_2$,

$$\begin{aligned} &\phi^{\gamma'}(\mu, k) \\ &= \phi^{\sigma^{l_1+l'_1-k_1-k'_1+k_2+k'_2}(\gamma')}(\phi^{\gamma'}(\mu, l_1 + l'_1 - k_1 - k'_1 + k_2 + k'_2), k - l_1 - l'_1 + k_1 + k'_1 - k_2 - k'_2) \\ &\overset{(132),(134)}{=} \phi^{\alpha''}(\mu'', k - l_1 - l'_1 + k_1 + k'_1 - k_2 - k'_2), \end{aligned}$$

and the last statement implies that

$$\omega^{\gamma'}(\mu) \overset{Theorem\ 3.3}{=} \omega^{\alpha''}(\mu'').$$

If $\omega^{\alpha''}(\mu'') \subset M_{i_1}$, then we have $M_{i_1} \perp M_{i_2}$ and this is a contradiction with the supposition that M_{i_1}, M_{i_2} are maximal and distinct. We can suppose without losing the generality that

$$\omega^{\alpha''}(\mu'') \subset M_{i_3}.$$

...

The reasoning makes the supposition that $M_{i_1}, ..., M_{i_{q-1}}$ are terminal nonfinal (pseudo-final) and that

$$\omega^{\alpha'''}(\mu''') \subset M_{i_q}.$$

If $\forall \alpha'''' \in \Pi_n, \forall \mu'''' \in M_{i_q}, O^{\alpha''''}(\mu'''') \subset M_{i_q}$, then M_{i_q} is final and the theorem is proved, otherwise $\alpha'''' \in \Pi_n, \mu'''' \in M_{i_q}$ exist such that $O^{\alpha''''}(\mu'''') \setminus M_{i_q} \neq \emptyset$, thus $\omega^{\alpha''''}(\mu'''') \wedge M_{i_q} = \emptyset$. If $\omega^{\alpha''''}(\mu'''') \subset M_{i_1}$, we get the contradiction $M_{i_1} \perp M_{i_q}$; if $\omega^{\alpha''''}(\mu'''') \subset M_{i_2}$, this implies the contradiction $M_{i_2} \perp M_{i_q}$; ... and if $\omega^{\alpha''''}(\mu'''') \subset M_{i_{q-1}}$, then the contradiction $M_{i_{q-1}} \perp M_{i_q}$ follows.

These are all the possibilities. One of $M_{i_1}, ..., M_{i_q}$ is final. ■

12 Definition of speed independence

Theorem 12.1 *For* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$ *and the point* $\mu \in \mathbf{B}^n$, *the following statements are equivalent:*

(a) the final set $A \in F^{\Phi}$ *exists such that*

$$\forall \alpha \in \Pi_n, O^{\alpha}(\mu) \wedge A \neq \emptyset,$$

(b) the final set $A \in F^{\Phi}$ *exists satisfying*

$$\forall \alpha \in \Pi_n, \omega^{\alpha}(\mu) \subset A, \tag{135}$$

(c) $\omega^{+}(\mu) \in \Omega_{\Phi}$,
(d) $\omega^{+}(\mu) \in F^{\Phi}$,
(e) $\exists \delta \in \Pi_n$,

$$\forall \lambda \in \mathbf{B}^n, \Phi^{\lambda}(\omega^{\delta}(\mu)) \subset \omega^{\delta}(\mu) \text{ and } \forall \alpha \in \Pi_n, \omega^{\delta}(\mu) \wedge \omega^{\alpha}(\mu) \neq \emptyset. \tag{136}$$

Proof. (a)$\Longrightarrow$(b) We take $\alpha \in \Pi_n$ arbitrary and we know that $k' \in \mathbf{N}$ exists with

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k) | k \geq k'\}.$$

The hypothesis states the existence of $k'' \in \mathbf{N}$ such that $\phi^{\alpha}(\mu, k'') \in A$. We use the invariance of A and we get that $\forall k \geq k'', \phi^{\alpha}(\mu, k) \in A$ thus, for $k_1 = \max\{k', k''\}$, we have

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k) | k \geq k_1\} \subset A.$$

(b)$\Longrightarrow$(c) The nonempty set $A \subset \mathbf{B}^n$ exists, which is final: $\exists \gamma \in \Pi_n, \exists \mu' \in \mathbf{B}^n$,

$$A = \omega^{\gamma}(\mu'), \tag{137}$$

$$\forall \lambda \in \mathbf{B}^n, \Phi^{\lambda}(A) \subset A,$$

and we have also

$$\omega^{+}(\mu) \overset{(135)}{\subset} A. \tag{138}$$

We prove

$$A \subset \omega^{+}(\mu). \tag{139}$$

Let $\alpha \in \Pi_n$ arbitrary, for which we get that

$$\omega^{\alpha}(\mu) \subset \omega^{+}(\mu) \overset{(138)}{\subset} A \overset{(137)}{=} \omega^{\gamma}(\mu'), \tag{140}$$

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k) | k \geq k'\},$$

$$\omega^{\gamma}(\mu') = \{\phi^{\gamma}(\mu', k) | k \geq k''\}$$

hold, with $k' \in \mathbf{N}, k'' \in \mathbf{N}$ suitably chosen. We infer the existence of $k_1 \geq k'$ and $k_2 \geq k''$ with

$$\phi^{\alpha}(\mu, k_1) \overset{(140)}{=} \phi^{\gamma}(\mu', k_2). \tag{141}$$

We define $\delta \in \Pi_n$ as

$$\delta^k = \begin{cases} \alpha^k, \ if \ k \in \{0, ..., k_1 - 1\}, \\ \gamma^{k - k_1 + k_2}, \ if \ k \geq k_1 \end{cases} \tag{142}$$

and we have $\forall k \in \{0, ..., k_1 - 1\}$,

$$\phi^{\delta}(\mu, k) \overset{(142)}{=} \phi^{\alpha}(\mu, k),$$

$$\phi^{\delta}(\mu, k_1) = \phi^{\alpha}(\mu, k_1) \overset{(141)}{=} \phi^{\gamma}(\mu', k_2), \tag{143}$$

$\forall k \geq k_1$,

$$\phi^{\delta}(\mu, k) = \phi^{\sigma^{k_1}(\delta)}(\phi^{\delta}(\mu, k_1), k - k_1) \tag{144}$$

$$\overset{(142),(143)}{=} \phi^{\sigma^{k_2}(\gamma)}(\phi^{\gamma}(\mu', k_2), k - k_1) = \phi^{\gamma}(\mu', k - k_1 + k_2).$$

We obtain, like previously, that

$$\omega^{\delta}(\mu) \overset{(144)}{=} \omega^{\gamma}(\mu'), \tag{145}$$

wherefrom

$$A = \omega^{\gamma}(\mu') = \omega^{\delta}(\mu) \subset \omega^{+}(\mu).$$

(139) holds.

We conclude that $\omega^{+}(\mu) = A = \omega^{\gamma}(\mu') \in \Omega_{\Phi}$.

(c)$\Longrightarrow$(d) The set $\omega^{+}(\mu)$ is terminal and, from Theorem 7.1, it is also final.

(d)$\Longrightarrow$(e) The fact that $\omega^{+}(\mu)$ is final means the existence of $\gamma \in \Pi_n, \mu' \in \mathbf{B}^n$ with the properties

$$\omega^{+}(\mu) = \omega^{\gamma}(\mu'), \tag{146}$$

$$\forall \lambda \in \mathbf{B}^n, \Phi^{\lambda}(\omega^{+}(\mu)) \subset \omega^{+}(\mu). \tag{147}$$

Let $\alpha \in \Pi_n$ arbitrary, fixed. We get the existence of $k' \in \mathbf{N}, k'' \in \mathbf{N}$ with

$$\omega^{\alpha}(\mu) = \{\phi^{\alpha}(\mu, k) | k \geq k'\},$$

$$\omega^{\gamma}(\mu') = \{\phi^{\gamma}(\mu', k) | k \geq k''\}.$$

From the fact that $\omega^{\alpha}(\mu) \overset{(146)}{\subset} \omega^{\gamma}(\mu')$ we have the existence of $k_1 \geq k', k_2 \geq k''$ such that

$$\phi^{\alpha}(\mu, k_1) = \phi^{\gamma}(\mu', k_2). \tag{148}$$

We define $\delta \in \Pi_n$ by (142) and we infer

$$\omega^{\delta}(\mu) \overset{(145)}{=} \omega^{\gamma}(\mu').$$

In other words if $\omega^{+}(\mu) \in F^{\Phi}$, then δ exists with $\omega^{+}(\mu) = \omega^{\delta}(\mu)$. Finally for any $\alpha' \in \Pi_n$,

$$\omega^{\delta}(\mu) \wedge \omega^{\alpha'}(\mu) = \omega^{+}(\mu) \wedge \omega^{\alpha'}(\mu) = \omega^{\alpha'}(\mu) \neq \emptyset.$$

(e)$\Longrightarrow$(a) $\delta \in \Pi_n$ exists such that the set $A = \omega^{\delta}(\mu)$ is final, and in addition, for any $\alpha \in \Pi_n$, we get

$$O^{\alpha}(\mu) \wedge A \supset \omega^{\alpha}(\mu) \wedge \omega^{\delta}(\mu) \overset{hyp}{\neq} \emptyset.$$

■

Definition 12.1 *The system* Φ *for which one of the previous properties (a), ..., (e) from Theorem 12.1 is true is called* ***speed independent with respect to*** $\mu \in \mathbf{B}^n$.

Remark 12.1 *The speed independence of* Φ *with respect to* μ *represents that special case when equation*

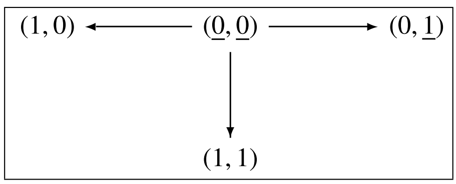
$$\omega^{+}(\mu) = M_{i_1} \vee M_{i_2} \vee ... \vee M_{i_p} \tag{149}$$

from Theorems 7.2 and 11.1 with $M_{i_1}, ..., M_{i_q}$ *maximal omega limit sets, at least one of which is final, becomes* $\omega^{+}(\mu) = M$*, where M is final.*

Remark 12.2 *We give from [1] the following citations concerning speed independence. 'Of special interest are those circuits in which the ultimate behavior of the circuit does not depend on the relative speeds of the elements. Such circuits, which will be called speed independent, may be designed without regard to time tolerances ... of elements and wiring. Hence they should be easier to design and more reliable than asynchronous circuits which require time tolerances on the elements for proper operation.' And later: 'we interpret the rather loose concept of ultimate behavior as meaning a specification of which terminal set is attained by an allowed sequence[2]. Thus if all allowed sequences starting with μ have the same terminal set we mean that circuit will always arrive, ultimately, at a unique static or dynamic condition.'*

13 Examples

Example 13.1 *The next system*



is not speed independent with respect to $\mu = (0, 0)$ *as far as in equation*

$$\omega^+(0, 0) = \{(1, 0)\} \vee \{(1, 1)\} \vee \{(0, 1)\}$$

three final sets occur, $\{(1, 0)\}, \{(1, 1)\}$ *and* $\{(0, 1)\}$.

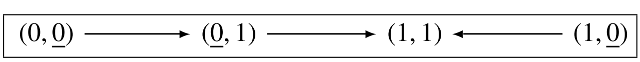
Example 13.2 *The identity* $1_{\mathbf{B}^n} : \mathbf{B}^n \to \mathbf{B}^n$ *is speed independent with respect to any* $\mu \in \mathbf{B}^n$ *because* $\omega^+(\mu) = \{\mu\}$ *is final.*

Example 13.3 *The constant function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, *for which* $\mu' \in \mathbf{B}^n$ *exists such that* $\forall \mu \in \mathbf{B}^n, \Phi(\mu) = \mu'$ *is speed independent with respect to any* μ *as far as the set* $\omega^+(\mu) = \{\mu'\}$ *is final.*

Example 13.4 *More general than previously, if* $\mu' \in \mathbf{B}^n$ *is a fixed point* $\Phi(\mu') = \mu'$ *that fulfills*

$$\forall \alpha \in \Pi_n, \forall \mu \in \mathbf{B}^n, \exists k' \in \mathbf{N}, \forall k \geq k', \phi^\alpha(\mu, k) = \mu'$$

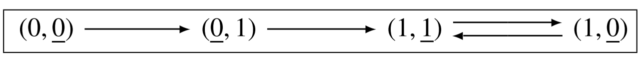
(μ' is called a global attractor in this case), the system Φ *is speed independent with respect to any* μ, *even if it is not the constant function equal with* μ'. *We give the example of such a system where* $\mu' = (1, 1)$.



Example 13.5 *Even more general than previously, we have the next possibility. The nonempty set* $A \subset \mathbf{B}^n$ *is final and*

$$\{\mu | \mu \in \mathbf{B}^n, \forall \alpha \in \Pi_n, \omega^\alpha(\mu) \subset A\} = \mathbf{B}^n$$

holds (such an A is said to be totally attractive). Then Φ *is speed independent with respect to any* μ *and* $\omega^+(\mu) = A$. *Here is an example for this situation*

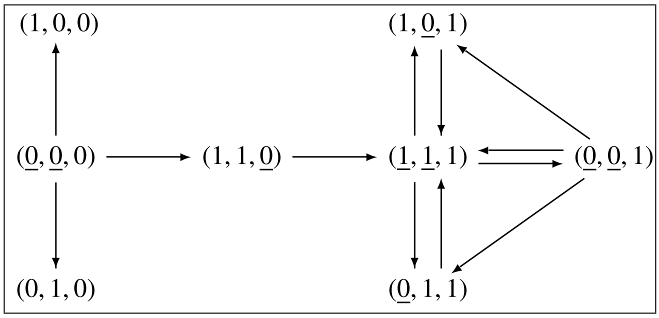


the system is speed independent with respect to any μ *and* $A = \{(1, 1), (1, 0)\}$.

[2] An allowed sequence is here a state function $\phi^\alpha(\mu, \cdot)$.

Example 13.6 *The function* $\Phi : \mathbf{B}^n \to \mathbf{B}^n$, $\forall \mu \in \mathbf{B}^n, \Phi(\mu) = \overline{\mu}$ *is also speed independent with respect to any* μ *and* $\forall \mu \in \mathbf{B}^n$, *the set* $\omega^+(\mu) = \mathbf{B}^n$ *is final.*

Example 13.7 *The system*



is not speed independent with respect to $(0, 0, 0)$, *but it is speed independent with respect to* $(1, 1, 0)$, *since* $\omega^+(1, 1, 0) = \{(1, 1, 1), (1, 0, 1), (0, 0, 1), (0, 1, 1)\}$ *is a final set.*

References

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[2] Serban E. Vlad, Boolean Systems: Topics in Asynchronicity, Academic Press, 2023 (to appear).

[3] Alexandre Yakovlev, Luciano Lavagno, Alberto Sangiovanni-Vincentelli, A unified signal transition graph model for asynchronous control circuit synthesis. Form Method Syst Des 9, 139–188 (1996). https://doi.org/10.1007/BF00122081.