Intensity variability in stationary solutions of the Fractional Nonlinear Schrödinger Equation

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Abstract. Solitons that propagate in optical fiber with indexes of refraction, dispersion, and diffraction are balanced, making pulses or electromagnetic waves propagate without any distortion. This is closely related to use of nonlinear refractive index in fiber optics. If an optical fiber only uses a nonlinear refractive index, then the partial signal can be lost over time. This study aims to analyze the variability of stationary solutions in multi-solitons formed using Fractional Nonlinear Schrödinger (FNLS). The parameter p indicates energy level of the solution to FNLS equation which has a positive integer value. This study focuses on 3 variations of p values, namely p = 0 which indicates the ground state, p = 1 which indicates the first excited state, and p = 2 which indicates the second excited state. During the first to second excited state, multi soliton peaks are formed with the same amplitude symmetrically. The amplitude experienced by the middle soliton in second excited state is lower which indicates the input signal obtained from the FNLS solution in the ground state in the form of triple-soliton. The polarization mode cause the soliton pulse width to shrink and the consequent amplitude in the first excited state to increase.

1 Introduction

Soliton is essentially a nonlinear wave or solitary wave that propagates within a nonlinear dispersive medium. A soliton is visually a continuous, stable mound of energy that occupies finite space, does not disperse, and moves at a fixed speed during propagation. John Scott Russell (1808-1882) was the first scientist to be credited with developing the soliton concept [1]. He proposed the soliton while watching the phenomena of water waves in the Edinburgh-Glasgow channel. Water waves flowing along the channel do not change shape at reasonably long intervals [2]. Norman Zabusky and Martin Kruskal coined the term "soliton" when investigating solitary waves on the Korteweg-de Vries equation (KdV) [3]. In general, solitons can be differentiated by their interaction with solitons' collision qualities and the number of solitons kept [4]. Solitary waves, on the other hand, do not suffer this. This is due to the fact that solitary waves, such as solitons, possess particle qualities. As a result, all solitary waves are solitons in general [5]. For the past fifty years, soliton has been the subject

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of theoretical and empirical research. Solitons have been explored in a variety of fields, including optics, plasmas, fluids, and many more [6].

Research on optical signal processing is an important aspect of optical communication systems because it has a very fast response rate [7]. This is closely related to the use of nonlinear refractive indices in optical fibers. If an optical fiber uses only a nonlinear refractive index, then the partial signal can be lost over time. This is due to the non-uniform distribution of intensity in pulses [8]. Improving system performance can be done by utilizing the natural properties possessed by optical fibers, namely Group Velocity Dispersion (GVD) and Self-Phase Modulation (SPM). The balance of GVD and SPM on an optical fiber can form soliton. Soliton can improve system performance due to its stable propagation along the fiber. The existence of solitons in optical fibers has been observed by experimental observations [9] and theoretical studies [10]. The propagation of solitons in optical fibers is theoretically approximated by an equation often called the Nonlinear Schrödinger (NLS) equation [11].

The NLS equation is an equation that can generally be used to observe the dynamics of waves propagating in a nonlinear dispersive medium, such as Bose-Einstein condensation [12–13], plasma physics [14], double-stranded deoxyribonucleic acid [15], electrical trajectories [16], fluid dynamics [17], and many more. In nonlinear optics, the NLS equation can be used to observe the propagation of electromagnetic waves in a single-mode [17–19] optical fiber medium. The NLS equation describes the propagation of electromagnetic waves within an optical fiber that changes slowly. This paper aims to model the propagation of electromagnetic waves in optical fibers based on the FNLSCQ equation based on the stationary solution.

2 Method

In general, NLS equations can be written in the form:

$$i\frac{d\Psi}{dt} + C\nabla^2\Psi - V\Psi + F(\Psi) = 0, \qquad (1)$$

where $\Psi \equiv \Psi(x, y, t) \in C$ is the wave function at $t \in R+$, $x, y \in R$, C is a real-valued parameter ($C \in R$). The term $F(\Psi)$ is a nonlinear term that has several forms, including [19]:

1. cubic type = $|\Psi|^2\Psi$
2. quintic type = $|\Psi|^4\Psi$
3. Ablowitz-Ladic type = $1/2\ |\Psi|^2\ \partial\Psi/\partial x$

Equation 1 is then converted into the Schrödinger Cubic Quintic Nonlinear Fractional Equation (FNLSCQ) which is capable of describing multi-soliton. The equation used in this study can be written in the form [20],

$$i\frac{d\Psi}{dt} - \frac{\beta}{2}(-\Delta)^{\frac{\rho}{2}}\Psi + \gamma|\Psi|^2\Psi + \sigma|\Psi|^4\Psi = V(x)\Psi + i\frac{\alpha}{2}\Psi + i\frac{\alpha^2}{6}\Psi^3 \qquad (2)$$

The stationary solution of the FNLSCQ equation is determined by selecting the value of the parameter. This is done by assuming that the influence of parameters is only experienced by waves when propagating in optical fiber (where stationary solutions are input waves that will later be propagated in optical fiber). In this study, the stationary solution of the FNLSCQ equation was determined by the NR method. The NR method can be applied to the FNLSCQ equation by entering the ansatz of the FNLSCQ equation written in the form:

$$\Psi(x,t) = \psi(x)e^{i\Omega_n t} \qquad (3)$$

The stability of the solution of the FNLSCQ equation can be done by adding a perturbation factor so that it can be written in the form,

$$\Psi(x,t) = \left(\psi(x) + \varepsilon\varphi(x,t) + i\varepsilon\phi(x,t) + \mathcal{O}(\varepsilon)\right)e^{-i\Omega_n t}, \quad (4)$$

where is a constant of very small value.$\varepsilon \in \mathbb{R}(\varepsilon \ll 1)$

In this study, it is choosed a different initial guess in each case. For the case of the FNLSCQC equation with, an initial condition of the form is selected, $V(x) = 0$

$$\psi_0(x) = \sqrt{2}\,\mathrm{sech}\left(\sqrt{2}x\right). \quad (5)$$

The initial guess for this case was selected based on FNLSCQ's standard equation analytic solution. Using the initial guess, we will find a stationary solution of the FNLSCQ equation for some value. As for the case, this study uses $(\alpha = 2, \beta = 1, \gamma = 1)\alpha V(x) = x^2$an initial guess in the form of,

$$\psi_{0_n}(x) = \frac{A(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}}\frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right), \quad (6)$$

with n is the energy level.

The dynamics of the solution of the FNLSCQ equation will be observed by the RK4 method. The RK4 method is one of the numerical methods that can be used to determine the solution of Ordinary Differential Equations (ODE) with a fairly small accuracy. It is characterized by a high order on the RK method. The RK4 method can be used with the form of equations,

$$y_{i+1} = y_i + hf(x_i, y_i) \quad (7)$$

by using the PDB in Equation 7, the RK4 method can be applied by,

$$\begin{aligned} k_1 &= hf(x_i, y_i) \\ k_2 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right) \\ k_4 &= hf(x_i + h, y_i + k_3) \\ y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \quad (8)$$

The parameter values to be used in this study have been presented in Table 1.

Table 1. NLS equation simulation parameters

Parameters	**Symbol**	**Value**
Group wave dispersion	β	$\sqrt{2}$
3rd order	γ	1
5th-order	σ	0.1
Lèvy Index	ρ	$[1,2] \in \mathbb{R}$
Attenuation	α	$[0,0.1] \in \mathbb{R}$
Energy (eigenvalue)	Ω_n	$1 + 2n$
Energy levels	n	$[0,2] \in \mathbb{Z}$
Initial guess parameters	A	$2(1 + n)$

3 Results and Discussion

The full equation of FNLS used in this study can be written as follows,

$$i\frac{\partial\Psi}{\partial t} - \frac{\beta}{2}(-\Delta)^{\frac{\rho}{2}}\Psi + \gamma|\Psi|^2\Psi + \sigma|\Psi|^4\Psi = \mathrm{V}(x)\Psi + i\frac{\alpha}{w}\Psi + i\frac{\alpha^2}{6}\Psi^3 \quad (9)$$

The stationary solution in this study was determined using the NR method, where the initial guess used in this study was in the form of,

$$\Psi_p(x) = \frac{2(1+p)(-1)^p}{\sqrt{2^p p!\sqrt{\pi}}}\frac{d^p}{dx^p}\left(e^{-\frac{x^2}{2}}\right) \tag{10}$$

p in equation 10 above indicates the energy level of the solution of the FNLS equation which is positive interger. The change in the p-value will affect the number of soliton peaks obtained from the solution of the FNLS equation. This study focused on 3 variations in p values, namely p = 0 which indicates the ground state, p = 1 which indicates the first excited statep $p \in \mathbb{Z}^+$, and p = 2 which indicates the second excited state. The stationary solution of the FNLS equation for the ground state (p = 0) is obtained as shown in Fig. 1.

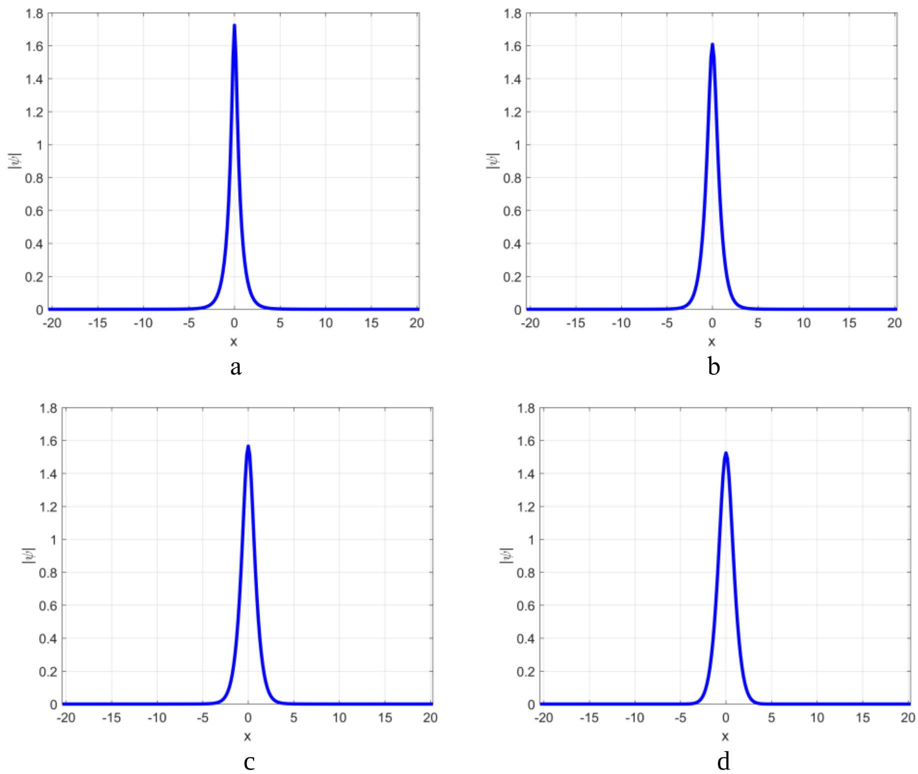


Fig. 1. Stationary solution of the FNLS equation for p = 0

Fig. 1 represents the stationary solution of the FNLS equation at ground state (p = 0) for the variation in values ($\rho(\rho \in (1; 2])$. It can be seen in Fig. 1 that at the ground state, only 1 soliton peak is formed which indicates the input signal obtained from the FNLS solution in the ground state in the form of single-soliton. The change in value leads to a narrowing of the pulse width of the soliton and an increase in the resulting amplitude. This indicates that the stronger the polarization mode (< 2), causing shrinkage in the pulses of electromagnetic waves propagated into the optical fiber. The stationary solution of the FNLS equation for the first excited state (p = 1) is obtained as shown in Fig. 2.

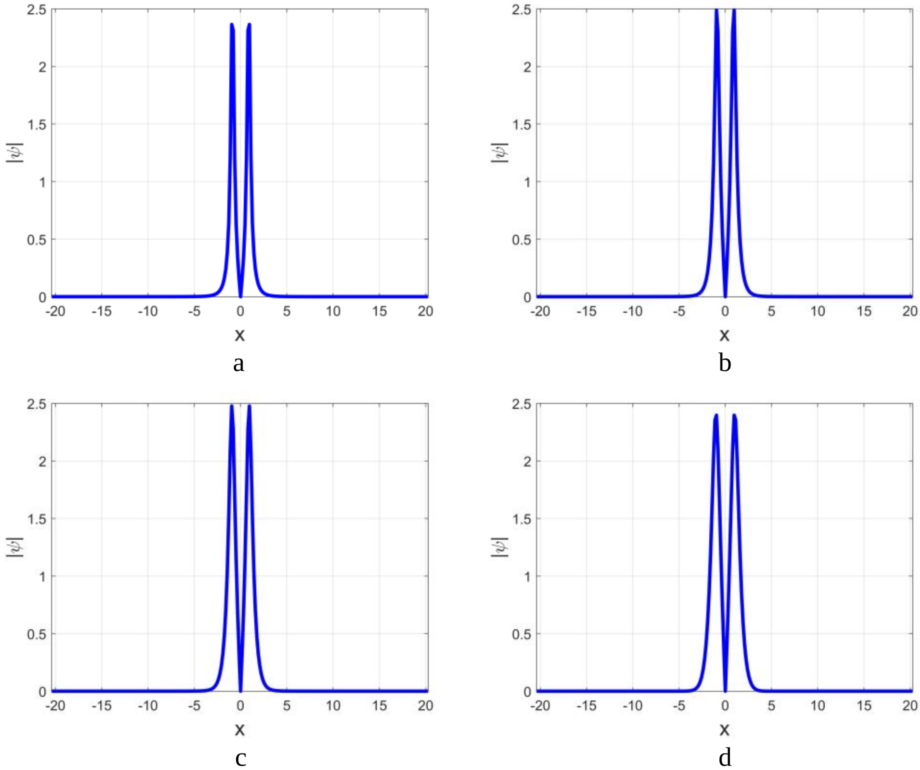
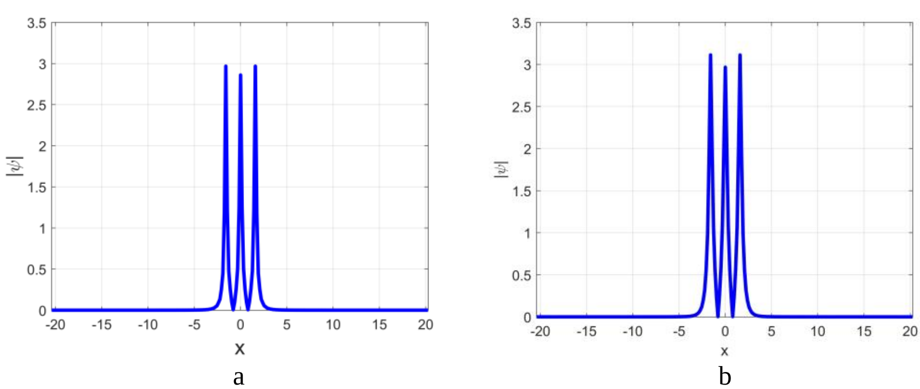


Fig. 2. Stationary solution of the FNLS equation for p = 1

Fig. 2 represents the stationary solution of the FNLS equation in the first excited state (p = 1) for the variation in values $\rho(\rho \in (1;2])$. It can be seen in Fig. 2 that at the time of the first excited state, 2 soliton peaks with the same amplitude are formed which indicates the input signal obtained from the FNLS solution in the ground state is dual-soliton. As experienced by stationary solutions in the ground state, changes in values lead to narrowing of the pulse width of the soliton and an increase in the resulting amplitude. This indicates that the stronger the polarization mode (< 2), causing shrinkage in the pulses of electromagnetic waves propagated into the optical fiber. The stationary solution of the FNLS equation for the second excited state (p = 2) is obtained as shown in Fig. 3.



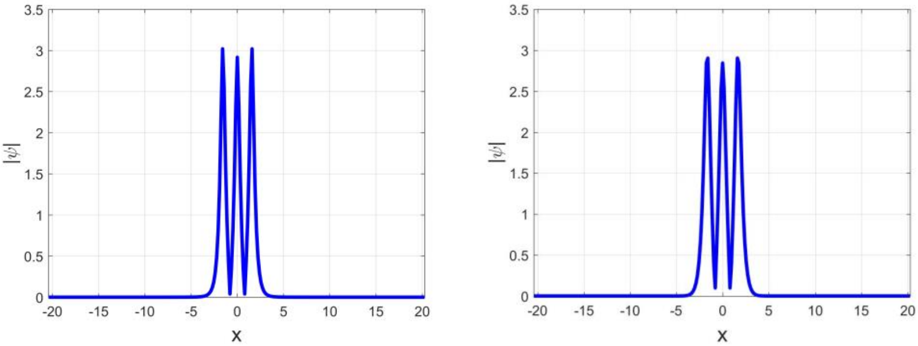


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Fig. 3. Stationary solution of the FNLS equation for p = 2

Fig. 3 represents the stationary solution of the FNLS equation in the first excited state (p = 2) for the variation in values $\rho(\rho \in (1; 2])$. It can be seen in Fig. 3 that during the second excited state, 3 soliton peaks are formed with the same amplitude symmetrically but the amplitude experienced by the middle soliton is lower which indicates the input signal obtained from the FNLS solution in the ground state in the form of triple-soliton. As experienced by stationary solutions in the ground state and the first excited state, the change in value causes a narrowing of the pulse width of the soliton and an increase in the resulting amplitude. This indicates that the stronger the polarization mode (< 2), causing shrinkage in the pulses of electromagnetic waves propagated into the optical fiber.

4 Conclusion

The stationary solution above indicates the energy level of the FNLS equation solution with a positive integral value. In the ground state, just one soliton peak is formed, suggesting that the input signal collected from the FNLS solution in the ground state is a single-soliton. When the value of decreases, the soliton pulse width narrows and the amplitude increases. This indicates that when the polarization mode becomes stronger, less electromagnetic wave pulse is propagated into the optical cable. During the first excited state, two soliton peaks with the same amplitude form, indicating that the input signal produced from the FNLS solution in the ground state is a dual-soliton. Changes in the value of p, as with stationary solutions in the ground state, cause the soliton pulse width to shorten and the consequent amplitude to increase. This indicates that the greater the polarization mode, the fewer electromagnetic wave pulses that enter the optical fiber. Three soliton peaks with the same amplitude arise symmetrically during the second excited state, but the soliton in the middle has a lower amplitude, indicating that the input signal obtained from the FNLS solution in the ground state is a triple-soliton. Changes in the value of polarization mode cause the soliton pulse width to shrink and the consequent amplitude in the first excited state to increase.

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