Modular irregularity strength of disjoint union of cycle-related graph

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Abstract. Let $G = (V, E)$ be a graph with a vertex set V and an edge set E of G, with order n. Modular irregular labeling of a graph G is an edge k-labeling $\varphi: E \rightarrow \{1, 2, \cdots, k\}$ such that the modular weight of all vertices is all different. The modular weight is defined by $wt_\varphi(u) = \sum_{v \in N(u)} \varphi(uv) \pmod n$. The minimum number k such that a graph G has modular irregular labeling with the largest label k is called modular irregularity strength of G. In this research, we determine the modular irregularity strength for a disjoint union of cycle graph, $\text{ms}(mC_n) = \frac{mn}{2} + 1$ for $n \equiv 0 \pmod 4$, a disjoint union of sun graph, $\text{ms}(m(C_n \odot K_1)) = \infty$ for n and m even and $\text{ms}(m(C_n \odot K_1)) = mn$ otherwise, and a disjoint union of middle graph of cycle graph, $\text{ms}(mM(C_n)) = \infty$ for n and m both odd numbers and $\text{ms}(mM(C_n)) = \frac{mn}{2} + 1$ otherwise.

Keywords. Modular irregularity strength, disjoint union of graph, cycle graph, sun graph, middle graph of cycle graph

1 Introduction

Graph labeling is a mapping from a set of numbers to elements of a graph G, usually the vertices or the edges [1]. Over the years, many graph labelings have been introduced; among them are irregular labeling and modular irregular labeling.

In [2], Chartrand et al. introduced irregular labeling. An edge labeling $\varphi: E(G) \rightarrow \{1,2, \dots, k\}$, where k is a positive integer, such that the weighst of the vertices are all different, is called irregular labeling. The weight of a vertex $u \in V(G)$ is defined by $wt_\varphi(u) = \sum_{v \in N(u)} \varphi(uv)$, where $N(u)$ denotes the set of neighbors of u in G. The irregularity strength of G, notated as $\text{s}(G)$ is the minimum number k for which the graph G has irregular labeling with label at most k. Chartrand et al. also give the lower bound of the irregularity strength of a graph as follows.

Theorem 1 [2]. Let G be a connected graph of order ≥ 3 containing n_i vertices of degree i, then,

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$$s(G) \geq \max_{1 \leq i \leq \Delta(G)} \left\{ \frac{n_i + i - 1}{i} \right\} \quad (1)$$

In [3] Bača et al. introduced modular irregular labeling as a variation of irregular labeling. Modular irregular labeling of a graph G, with order n, is an edge k-labeling $\varphi: E \rightarrow \{1, 2, \cdots, k\}$ such that the modular weights of all vertices are all different. The modular weight is defined by $wt_{\varphi}(u) = \sum_{v \in N(u)} \varphi(uv) \pmod n$, where $N(u)$ denotes the set of neighbors of u in G. The minimum number k such that a graph G has modular irregular labeling with the largest label k is called modular irregularity strength of G. If no modular irregular labeling for the graph G could be found, it is defined as $ms(G) = \infty$. The relation between $\mathrm{ms}(G)$ and $\mathrm{s}(G)$ is provided by Bača et al. in the following theorem.

Theorem 2 [3]. Let G be a graph with no component of order ≤ 2. Then, $\mathrm{s}(G) \leq \mathrm{ms}(G)$.

Bača et al. also characterized a sufficient condition such that a graph has no modular irregular labeling.

Theorem 3 [3]. If G is a graph with order $n, n \equiv 2 \pmod 4$, then G has no modular irregular labeling, that is, $\mathrm{ms(G)} = \infty$.

There are some classes of graphs for which the modular irregularity strength has been found. Bača et al. [3] determined the modular irregularity strength of some graphs, namely path, star, triangular, cycle, and gear graphs. Muthugurupackiam and Ramya determined the modular irregularity strength of tadpole and double-cycle graphs [4]. Bača et al. also determined the modular irregularity strength of the fan graph [5] and wheel [6]. Then Sugeng et al. determined the modular irregularity strength of double-star and friendship graphs [7]. Tilukay determined the modular irregularity strength of the triangular book graph [8]. Then Hinding et al. determined the modular irregularity strength of the dodecahedral-modified generalization graph [9]. Dewi determined the modular irregularity strength of $C_n \odot mK_1$ [10]. Recently, Sugeng determined the modular irregularity strength of $M(C_n)$ and several other flower-type graphs [11].

A cycle graph C_n is a connected graph with order $n \geq 3$ which has a vertex set $V(C_n) = \{v_1 : 1 \leq i \leq n\}$ and an edge set $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$ [12].

A sun graph $C_n \odot K_1$ is the graph obtained from a cycle graph C_n by adding a pendant edge to every vertex in the cycle [13]. Thus, $C_n \odot K_1$ has $2n$ vertices.

The middle graph $M(G)$ of a connected graph $G = (V(G), E(G))$ is defined as a graph where $V(M(G)) = V(G) \cup E(G)$ and two vertices u and v are adjacent if:

1. u and v are adjacent edges of G, or
2. u is a vertex of G, and v is an edge that incident with it , or vice versa [14].

The graph union of two graphs $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$ is the graph $G = G_1 \cup G_2$ whose vertex-set is the disjoint union of the vertex-sets of G_1 and G_2, that is $V(G) = V_1 \cup V_2$, and the edge-set is the disjoint union of the edge-set of G_1 and G_2, that is $E(G) = E_1 \cup E_2$. The iterated union $G \cup G \cup \ldots \cup G$ of n disjoint copies of the graph G notated as nG [15].

The $\mathrm{ms}(C_n)$, $\mathrm{ms}(C_n \odot mK_1)$, and $\mathrm{ms}(M(C_n))$ have been determined [3,10,11]. In this research, we determine the modular irregularity strength for a disjoint union of the cycle graph, a disjoint union of the sun graph, and a disjoint union of the middle graph of the cycle graph.

2 Results and discussion

In this section, we determined the modular irregularity strength a disjoint union of cycle graph, a disjoint union of sun graph, and a disjoint union of middle graph of cycle graph.

2.1 Disjoint union of cycle graph

Lemma 1. Let C_n be a cycle graph with $n \equiv 0 \pmod 4$ and mC_n be the disjoint union of m copies of C_n, $m \geq 1$. Then,

$$\mathrm{ms}(mC_n) \geq \frac{mn}{2} + 1.$$

Proof. According to Theorem 1, we obtain:

$$\mathrm{s}(mC_n) \geq \frac{mn-1}{2} + 1.$$
$$\mathrm{s}(mC_n) \geq \frac{mn+1}{2}.$$
$$\mathrm{s}(mC_n) \geq \frac{mn}{2} + 1.$$

Then, based on Theorem 2, we obtain:

$$\mathrm{ms}(mC_n) \geq \mathrm{s}(mC_n) \geq \frac{mn}{2} + 1.$$
$$\mathrm{ms}(mC_n) \geq \frac{mn}{2} + 1. \blacksquare$$

Theorem 4. Let C_n be a cycle graph with $n \equiv 0 \pmod 4$ and let mC_n be the disjoint union of m copies of C_n, $m \geq 1$. Then,

$$\mathrm{ms}(mC_n) = \frac{mn}{2} + 1.$$

Proof. Let $V(mC_n) = \{v_i^j : 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(mC_n) = \{v_i^j v_{i+1}^j : 1 \leq i \leq n; 1 \leq j \leq m\}$ be the vertex set and edge set of the graph mC_n respectively, where $v_{n+1}^j = v_1^j$. We define ϕ as the following edge labeling,

$$\phi(v_i^j v_{i+1}^j) = i + \frac{n}{2}(j-1), \qquad 1 \leq i \leq \frac{n}{2}, 1 \leq j \leq m.$$
$$\phi(v_{n+1-i}^j v_{n+2-i}^j) = 2\left\lfloor \frac{i}{2} \right\rfloor + 1 + \frac{n}{2}(j-1), \qquad 1 \leq i \leq \frac{n}{2}, 1 \leq j \leq m.$$

The weights of the vertices of mC_n under the labeling ϕ are $\{2,3,\dots,mn+1\}$. Then the modular weights of the vertices of mC_n under the labeling ϕ are

$\{0,1,\ldots,mn-1\}$, under the modulo mn. Therefore, ϕ fulfill the requirement of a modular irregular labeling for mC_n and we can conclude that,

$$\mathrm{ms}(mC_n) \leq \frac{mn}{2} + 1.$$

Hence, according to Equation (1) and Lemma 1, we can conclude that,

$$\mathrm{ms}(mC_n) = \frac{mn}{2} + 1. \blacksquare$$

The example of Theorem 4 for modular irregular labeling on the disjoint union of three cycle graph C_8 is shown in Fig. 1.

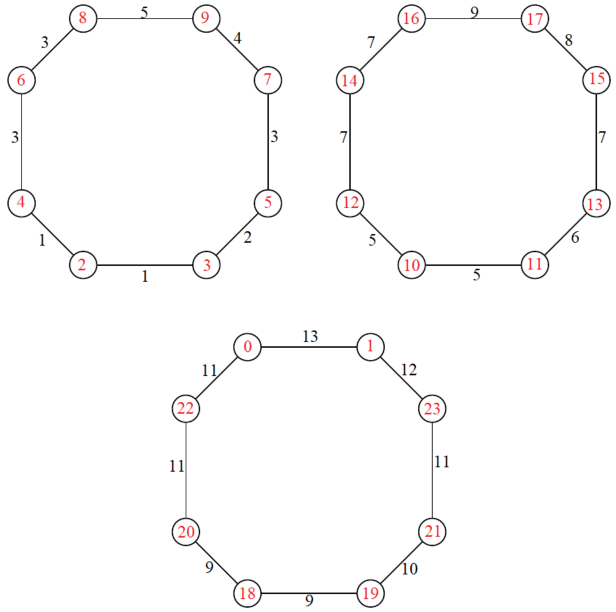

Fig. 1. Modular irregular labeling on disjoint union of three cycle graph C_8.

2.2 Disjoint union of sun graph

Theorem 5. Let $C_n \odot K_1$ be a sun graph and let $m(C_n \odot K_1)$ be the disjoint union of m copies of $C_n \odot K_1$, $m \geq 1$. Then,

$$\mathrm{ms}(m(C_n \odot K_1)) = \begin{cases} \infty, & n \text{ and } m \text{ odd}, \\ mn, & \text{otherwise}. \end{cases}$$

Proof. Let $V(m(C_n \odot K_1)) = \{a_i^j, b_i^j : 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(m(C_n \odot K_1)) = \{a_i^j a_{i+1}^j, a_i^j b_i^j : 1 \leq i \leq n; 1 \leq j \leq m\}$ be the vertex set and edge set of the graph $m(C_n \odot K_1)$ respectively, where a_i^j are the vertices from the cycle graph C_n and b_i^j are the pendant vertices. Let $a_{n+1}^j = a_1^j$. The proof will be divided into 2 cases:

- Case 1: n odd and m odd
  Let $n = 2k + 1$ and $m = 2l + 1$ for $n, l \in \mathbb{Z}$, from which we obtain:

$$|V| = 2(2l + 1)(2k + 1)$$
$$= 8kl + 4k + 4l + 2.$$
$$|V| = 4(2kl + k + l) + 2 \equiv 2(\text{mod } 4).$$

  Hence, according to Theorem 3, we can conclude that $\text{ms}(mM(C_n)) = \infty$.
- Case 2: n odd and m even or n even
  We define θ as the following edge labelling.

$$\theta(a_i^j a_{i+1}^j) = \frac{mn}{2}, \qquad 1 \le i \le n, 1 \le j \le m,$$

$$\theta(a_i^j b_i^j) = i + n(j-1), \qquad 1 \le i \le n, 1 \le j \le m.$$

  The weights of the vertices of $m(C_n \odot K_1)$ under the labeling θ are $\{1,2,\dots,2mn\}$. Then the modular weights of the vertices of $m(C_n \odot K_1)$ under the labeling ϕ are $\{0,1,\dots,2mn-1\}$, under the modulo $2mn$. Therefore, θ fulfill the requirement of a modular irregular labeling for $m(C_n \odot K_1)$, and we can conclude that,

$$\text{ms}(m(C_n \odot K_1)) \le mn.$$

  The pendant vertices must have different weights, so the minimum number of labels must be mn, that is,

$$\text{ms}(m(C_n \odot K_1)) \ge mn.$$

  Hence, we can conclude that,

$$\text{ms}(m(C_n \odot K_1)) = mn.$$

  From the cases above, we can conclude that,

$$\text{ms}(m(C_n \odot K_1)) = \begin{cases} \infty, & n \text{ and } m \text{ odd}, \\ mn, & \text{otherwise}. \end{cases} \blacksquare$$

The example of Theorem 5 for modular irregular labeling on the disjoint union of four sun graph $C_6 \odot K_1$ is shown in Fig. 2.

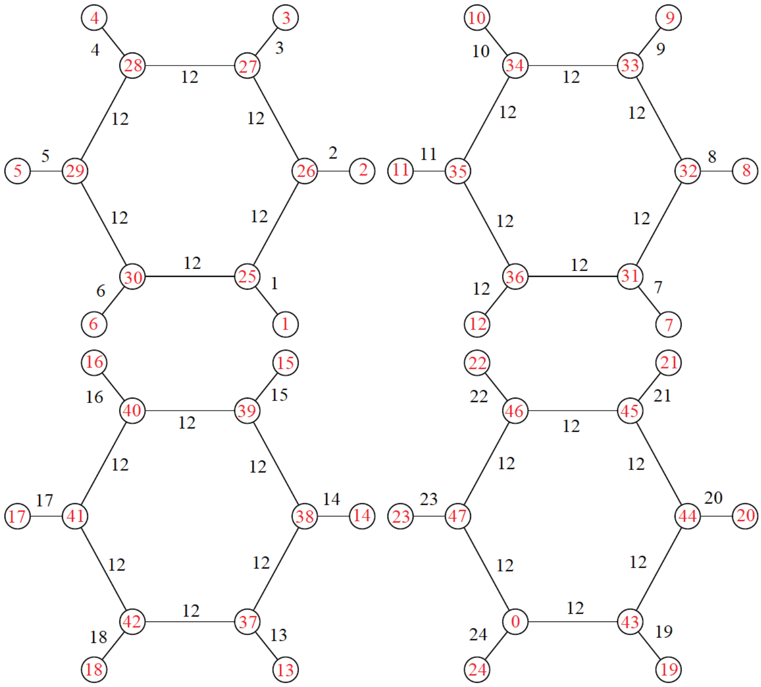

Fig. 2. Modular irregular labeling on disjoint union of four sun graph $C_6 \odot K_1$.

Lemma 2. Let $M(C_n)$ be a middle graph of cycle graph with $n \geq 3$ and let $mM(C_n)$ be the disjoint union of m copies of $M(C_n)$, $m \geq 1$. Then,

$$\mathrm{ms}(mM(C_n)) \geq \frac{mn}{2} + 1.$$

Proof. According to Theorem 1, we obtain:

$$\mathrm{s}(mM(C_n)) \geq \frac{mn-1}{2} + 1.$$

$$\mathrm{s}(mM(C_n)) \geq \frac{mn+1}{2}.$$

$$\mathrm{s}(mM(C_n)) \geq \frac{mn}{2} + 1.$$

Then, from Theorem 2, we obtain:

$$\mathrm{ms}(mM(C_n)) \geq \mathrm{s}\big(mM(C_n)\big) \geq \frac{mn}{2} + 1.$$

$$\mathrm{ms}(mM(C_n)) \geq \frac{mn}{2} + 1. \blacksquare$$

Theorem 6. Let $M(C_n)$ be a middle graph of cycle graph with $n \geq 3$ and let $mM(C_n)$ be the disjoint union of m copies of $M(C_n)$, $m \geq 1$. Then,

$$\mathrm{ms}\big(mM(C_n)\big) = \begin{cases} \infty, & n \text{ and } m \text{ odd}, \\ \dfrac{mn}{2} + 1, & \text{otherwise.} \end{cases}$$

Proof. Let $V(m(\,M(C_n))) = \{a_i^j, b_i^j : 1 \leq i \leq n; 1 \leq j \leq m\}$ and $E(m(\,M(C_n)\,)) = \{a_i^j a_{i+1}^j, a_i^j b_i^j, a_{i+1}^j b_i^j : 1 \leq i \leq n; 1 \leq j \leq m\}$ be the vertex set and edge set of the graph $m(\,M(C_n))$ respectively, where a_i^j are the vertices from the edges of cycle graph C_n and b_i^j are the vertices from the vertices of cycle graph C_n. Let $a_{n+1} = a_1$. The proof will be divided into 3 cases:

- Case 1: n even
  We define ψ as the following edge labelling,

$$\psi\big(a_i^j a_{i+1}^j\big) = \frac{mn}{2}, 1 \leq i \leq n;\ 1 \leq j \leq m.$$
$$\psi\big(a_1^j a_n^j\big) = \frac{mn}{2}, 1 \leq j \leq m.$$
$$\psi\big(a_i^j b_i^j\big) = \begin{cases} i + \dfrac{n}{2}(j-1), & 1 \leq i \leq \dfrac{n}{2} + 1; 1 \leq j \leq m, \\ n + 2 - i + \dfrac{n}{2}(j-1), & \dfrac{n}{2} + 2 \leq i \leq n; 1 \leq j \leq m. \end{cases} \quad (2)$$
$$\psi\big(a_{i+1}^j b_i^j\big) = \begin{cases} i + \dfrac{n}{2}(j-1), & 1 \leq i \leq \dfrac{n}{2}; 1 \leq j \leq m, \\ \dfrac{n}{2} + \dfrac{n}{2}(j-1), & i = \dfrac{n}{2} + 1; 1 \leq j \leq m, \\ n + 1 - i + \dfrac{n}{2}(j-1), & \dfrac{n}{2} + 2 \leq i \leq n; 1 \leq j \leq m. \end{cases}$$

The weights of the vertices of $mM(C_n)$ under the labeling ϕ are $\{2,3,\ldots,2mn+1\}$. Then the modular weights of the vertices of $mM(C_n)$ under the labeling ϕ are $\{0,1,\ldots,2mn-1\}$, under the modulo $2mn$. Therefore, θ fulfill the requirement of modular irregular labeling for $mM(C_n)$, and we can conclude that,

$$\mathrm{ms}\big(mM(C_n)\big) \leq \frac{mn}{2} + 1. \quad (3)$$

Hence, according to Equation (3) and Lemma 2, we can conclude that,

$$\mathrm{ms}\big(mM(C_n)\big) = \frac{mn}{2} + 1.$$

- Case 2: n odd and m even
  We define ψ as the following edge labelling,

$$\psi\big(a_i^j a_{i+1}^j\big) = \frac{mn}{2}, 1 \leq i \leq n;\ 1 \leq j \leq m.$$
$$\psi\big(a_1^j a_n^j\big) = \frac{mn}{2}, 1 \leq j \leq m.$$

$$\psi\left(a_i^j b_i^j\right)=\begin{cases} i+\dfrac{n(j-1)}{2}, & 1\le i\le \dfrac{n+1}{2}; j \text{ odd},\\ n+2-i+\dfrac{n(j-1)}{2}, & \dfrac{n+3}{2}\le i\le n; j \text{ odd},\\ \dfrac{n-1}{2}+i+\dfrac{n(j-2)}{2}, & 1\le i\le \dfrac{n+1}{2}; j \text{ even},\\ \dfrac{3n+3}{2}-i+\dfrac{n(j-2)}{2}, & \dfrac{n+3}{2}\le i\le n; j \text{ even}.\end{cases}$$

$$\psi\left(a_{i+1}^j b_i^j\right)=\begin{cases} i+\dfrac{n(j-1)}{2}, & 1\le i\le \dfrac{n+1}{2}; j \text{ odd},\\ n+1-i+\dfrac{n(j-1)}{2}, & \dfrac{n+3}{2}\le i\le n; j \text{ odd},\\ \dfrac{n+1}{2}+i+\dfrac{n(j-2)}{2}, & 1\le i\le \dfrac{n+1}{2}; j \text{ even},\\ \dfrac{3n+3}{2}-i+\dfrac{n(j-2)}{2}, & \dfrac{n+3}{2}\le i\le n; j \text{ even}.\end{cases}$$

The weight set of the vertices of $mM(C_n)$ under the labeling ϕ is $\{2,3,\dots,2mn+1\}$. Then the modular weights of the vertices of $mM(C_n)$ under the labeling ϕ are $\{0,1,\dots,2mn-1\}$, under the modulo $2mn$. Therefore, θ fulfill the requirement of a modular irregular labeling for $mM(C_n)$, and we can conclude that,

$$\mathrm{ms}\big(mM(C_n)\big)\le \frac{mn}{2}+1. \tag{4}$$

Hence, according to Equation (4) and Lemma 2, we can conclude that,

$$\mathrm{ms}\big(mM(C_n)\big)= \frac{mn}{2}+1.$$

- Case 3: n odd and m odd
  Let $n=2k+1$ and $m=2l+1$ for $n,l\in\mathbb{Z}$, from which we obtain:

$$\begin{aligned}|V|&=2(2l+1)(2k+1)\\ &=8kl+4k+4l+2.\\ |V|&=4(2kl+k+l)+2\equiv 2(\mathrm{mod}\ 4).\end{aligned}$$

Hence, according to Theorem 3, we can conclude that $\mathrm{ms}\big(mM(C_n)\big)=\infty$.
From the cases above, we can conclude that,

$$\mathrm{ms}\big(mM(C_n)\big)=\begin{cases}\infty, & n \text{ and } m \text{ odd},\\ \dfrac{mn}{2}+1, & \text{otherwise}.\end{cases} \quad ■$$

The example of Theorem 6 for modular irregular labeling on the disjoint union of three middle graphs of cycle graph C_6, $3M(C_6)$, is shown in Fig. 3.

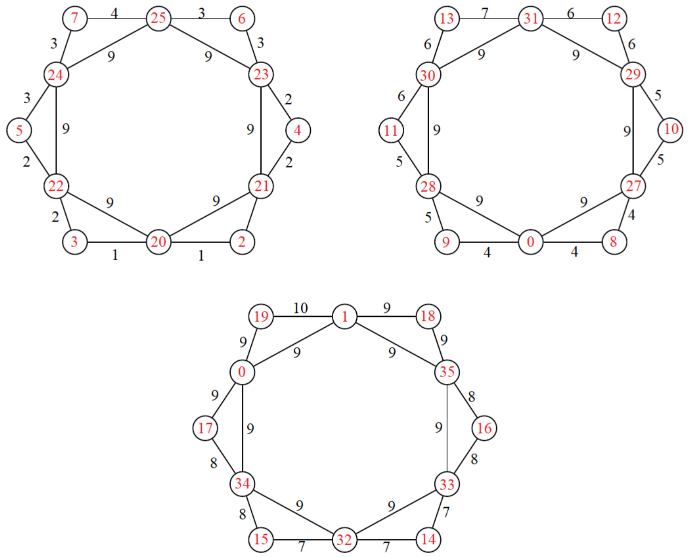

Fig. 3. Modular irregular labeling on disjoint union of three $M(C_6)$.

3 Conclusion

In this paper, we proved the modular irregularity strength of three classes of graphs. A disjoint union of the cycle graph has $\mathrm{ms}(mC_n) = \frac{mn}{2} + 1$ for $n \equiv 0 \pmod 4$. A disjoint union of the sun graph has $\mathrm{ms}(m(C_n \odot K_1)) = \infty$ for n and m even and $\mathrm{ms}(m(C_n \odot K_1)) = mn$ otherwise. Lastly, a disjoint union of the middle graph of the cycle graph has $\mathrm{ms}\big(mM(C_n)\big) = \infty$ for n and m both odd numbers, and $\mathrm{ms}\big(mM(C_n)\big) = \frac{mn}{2} + 1$ otherwise.

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