Simulations of Cayley graphs of dihedral group

Mohammad Farhan, *Peter* John, and *Denny Riama* Silaban*

Department of Mathematics, Faculty of Mathematics and Natural Sciences (FMIPA), Universitas Indonesia, Depok 16424, Indonesia

Abstract. Let Γ be a finite group with identity element e and let $S \subseteq \Gamma - \{e\}$ which is inverse-closed, i.e., $S = S^{-1} := \{s^{-1} : s \in S\}$. An undirected Cayley graph on a group Γ with connection set S, denoted by $\mathrm{Cay}(\Gamma, S)$, is a graph with vertex set Γ and edges xy for all pairs $x, y \in \Gamma$ such that $xy^{-1} \in S$. The dihedral group of order $2n$, denoted by D_{2n}, is a group generated by two elements a and b subject to the three relations: $a^n = e$, $b^2 = e$, and $bab^{-1} = a^{-1}$. In this paper, we investigate the Cayley graphs of the dihedral group for some connection sets S satisfying $|S| = 1, 2, 3$ by doing simulations using Wolfram Mathematica.

Keywords. Cayley graph, dihedral group, simulation

1 Introduction

A nonempty set Γ is said to be a group if in Γ there is defined an operation $*$ such that (1) if $a, b \in \Gamma$, then $a * b \in \Gamma$, (2) if $a, b, c \in \Gamma$, then $a * (b * c) = (a * b) * c$, (3) there exists an element $e \in \Gamma$ such that $a * e = e * a = a$ for all $a \in \Gamma$ (such element is called the identity element of Γ), and (4) for every $a \in \Gamma$ there exists an element $b \in \Gamma$ such that $a * b = b * a = e$ (b is called the inverse of a in Γ and b can be written as a^{-1}) [1]. One way to describe some groups is by giving generators and relations. An example of a group that can be described by generators and relations is the dihedral group. The dihedral group of order $2n$, denoted by D_{2n}, is a group generated by two elements a and b subject to the three relations: $a^n = e$, $b^2 = e$, and $bab^{-1} = a^{-1}$ where e is the identity element in D_{2n} [2]. The set D_{2n} can be written as $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$.

A graph $G = (V, E)$ is an ordered pair of a finite non-empty set V of objects called vertices and a set E of 2-element subsets of V called edges [3]. In this research, all graphs are simple and undirected. A Cayley graph is a graph that is constructed from a group using certain rules as follows. Let Γ be a finite group with identity element e and let $S \subseteq \Gamma$. Initially, a Cayley graph was defined as a directed graph, that is, a Cayley graph on Γ with a connection set S, denoted by $\mathrm{Cay}(\Gamma, S)$, is a graph with vertex set Γ and arcs (x, y) for all pairs $x, y \in \Gamma$ such that $xy^{-1} \in S$. If $e \notin S$ and S is inverse-closed, that is, $S = S^{-1} \coloneqq \{s^{-1} : s \in S\}$, then the Cayley graph is simple and undirected [4]. Therefore, in this research, we use S that satisfies $e \notin S$ and S is inverse-closed. The Cayley graph has an important role in constructing

* Corresponding author: denny@sci.ui.ac.id

expander graphs [5, 6] and its adjacency matrix eigenvalues have an important role in algebraic graph theory and have applications in various fields, such as chemical graph theory, quantum computing, and others [4]. On the other hand, Cayley graphs can also be defined on a semigroup [7] and ring [8, 9]. Another research on Cayley graphs can be found in [10, 11].

In this paper, we investigate the Cayley graphs of the dihedral group for some connection sets S satisfying $|S| = 1, 2, 3$ by writing a program in the Mathematica software to construct Cayley graphs of a dihedral group.

2 Materials and method

This paper was prepared by conducting literature reviews on Cayley graphs, dihedral groups, and their properties. Then, write a program in Mathematica to construct Cayley graphs $\mathrm{Cay}(D_{2n}, S)$ given the value of n and the connection set S. Then, find some connection sets for Cayley graphs of the dihedral group with cardinality 1, 2, and 3 and use them to construct Cayley graphs of the dihedral group.

3 Results and discussion

3.1 Isomorphism between dihedral group and $\mathbb{Z}_n \times \mathbb{Z}_2$

Let $\mathbb{Z}_n \times \mathbb{Z}_2$ be the Cartesian product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$, that is, $\mathbb{Z}_n \times \mathbb{Z}_2 \coloneqq \{(x, y) : x \in \mathbb{Z}_n, y \in \mathbb{Z}_2\}$. This discussion aims to construct an isomorphism between the dihedral group D_{2n} and $\mathbb{Z}_n \times \mathbb{Z}_2$ by constructing an operation in $\mathbb{Z}_n \times \mathbb{Z}_2$. This isomorphism will make it easier to write and analyze neighbors of a vertex in the following discussion.

Observe that every element in D_{2n} can be written in the form $a^x b^y$ with $x \in \mathbb{Z}_n$ and $y \in \mathbb{Z}_2$. We construct a mapping $f : D_{2n} \to \mathbb{Z}_n \times \mathbb{Z}_2$ that maps $a^x b^y \in D_{2n}$ to the ordered pair $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_2$, that is,

$$f(a^x b^y) = (x, y) \text{ for all } a^x b^y \in D_{2n}. \tag{1}$$

For example, $a^2 b$ maps to $(2,1)$, a maps to $(1,0)$, and identity element e maps to $(0,0)$.

We construct an operation in $\mathbb{Z}_n \times \mathbb{Z}_2$ defined by,

$$(x_1, y_1)(x_2, y_2) = \begin{cases} (x_1 + x_2, y_2), & y_1 = 0; \\ (x_1 - x_2, y_2 + 1), & y_1 = 1, \end{cases} \tag{2}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_n \times \mathbb{Z}_2$ with addition and subtraction in the first and second components are in modulo n and modulo 2, respectively. It can be proved that the set $\mathbb{Z}_n \times \mathbb{Z}_2$ with the operation defined in Equation (2) form a group and the mapping f defined in Equation (1) is an isomorphism between D_{2n} and $\mathbb{Z}_n \times \mathbb{Z}_2$. Therefore, the notations $a^x b^y$ and (x, y) are interchangeable.

3.2 Construction of Cayley graphs of dihedral group with Mathematica

Before we explain the construction process, it is necessary to note the following observation. Let Γ be a finite group and $S \subseteq \Gamma - \{e\}$ which is inverse-closed. Consider the Cayley graph $\mathrm{Cay}(\Gamma, S)$. Since $\mathrm{Cay}(\Gamma, S)$ is undirected and S is inverse-closed, the vertices x and y are adjacent if and only if $xy^{-1} \in S$ and $yx^{-1} \in S$. That is, there exists an $s \in S$ such that $yx^{-1} = s \Longrightarrow y = sx$. Then for each $x \in \Gamma$ and $s \in S$, the vertex x adjacent with the vertex sx and form the edge $\{x, sx\}$. Therefore, we obtain a technique to construct edges that are incident to a vertex. This technique is used in the construction of Cayley graphs of the dihedral group explained in the following.

Suppose we want to construct $\mathrm{Cay}(D_{2n}, S)$ using the program in Fig. 1. The inputs of this program are the value of n and the connection set S. Let f be the mapping defined in Equation (1). First, we define a function "Prod" that maps two elements $(x_1, y_1), (x_2, y_2) \in f(D_{2n})$ to the product $(x_1, y_1)(x_2, y_2)$ with the operation defined in Equation (2). Then, we construct the vertex set of $\mathrm{Cay}(D_{2n}, S)$ which is the set D_{2n} represented by $f(D_{2n}) = \mathbb{Z}_n \times \mathbb{Z}_2$, since their notations are interchangeable. Then we construct the edge set of $\mathrm{Cay}(D_{2n}, S)$ as follows. For each $(x, y) \in f(D_{2n})$ and $s \in S$, we construct the edge $\{(x, y), s(x, y)\}$. The output of this program is the figure of the Cayley graph $\mathrm{Cay}(D_{2n}, S)$ with the previously constructed vertex and edge sets. Note that in Mathematica, the tuple $(x, y) \in \mathbb{Z}_n \times \mathbb{Z}_2$ is written as $\{x, y\}$. Since this program runs iteratively, it will create the edges xy and yx for all x and y; hence, it will create multiple edges. Since Cayley graphs discussed are simple, these multiple edges will be reduced into a single edge.

```
CayleyDihedral[n_, S_] :=
 Module[{Prod, x, y, VG, EG},

  (* operation in Zn x Z2 *)
  Prod[{x1_, y1_}, {x2_, y2_}] := {Mod[x1 + (-1)^y1 * x2, n], Mod[y1 + y2, 2]};

  (* vertex set in Cay(D2n,S) *)
  VG := {};
  For[x = 0, x ≤ n - 1, x++,
   For[y = 0, y ≤ 1, y++,
    VG = Append[VG, {x, y}]
   ]
  ];

  (* edge set in Cay(D2n,S) *)
  EG := {};
  For[i = 1, i ≤ 2 n, i++,
   {x, y} = {VG[[i, 1]], VG[[i, 2]]};
   For[j = 1, j ≤ Length[S], j++,
    EG = Append[EG, {x, y} ⟷ Prod[S[[j]], {x, y}]];
   ]
  ];

  (* reduce multiple edges and produce the image of Cay(D2n,S) *)
  SimpleGraph[
   Graph[VG, EG], VertexLabels → Automatic, VertexSize → 0.2,
   VertexStyle → Black, EdgeStyle → Black
  ]
 ]
```

Fig. 1. Mathematica program to construct Cayley graphs of the dihedral group

3.3 Cayley graphs of dihedral group

This subsection gives the results of Cayley graphs of the dihedral group for some connection sets based on the construction program in Figure 1. This discussion is divided into three parts based on the cardinality of S, that is, $|S| = 1$, $|S| = 2$, and $|S| = 3$. We begin this discussion by giving an observation that will be helpful in some proofs discussed later.

Observation 1. Let $n \in \mathbb{Z}, n \geq 3$. Let D_{2n} be the dihedral group of order $2n$ and $e \in D_{2n}$ be the identity element. Let $S \subseteq D_{2n} - \{e\}$ which is inverse-closed and $s \in S$. Then s is self-inverse (that is, $s = s^{-1}$) if and only if $s = a^r b$ with $r \in \{0, 1, 2, \dots, n-1\}$ or $s = a^{n/2}$ if n is even.

Proof. If $s = a^r b$ with $r \in \{0, 1, 2, \dots, n-1\}$, then $s^{-1} = (a^r b)^{-1} = b^{-1}(a^r)^{-1} = b(a^{-r}) = a^r b = s$ since by the definition of dihedral group, $b^2 = e \Longrightarrow b = b^{-1}$ and $bab^{-1} = a^{-1} \Longrightarrow ba^{-1} = ab$ so that it can be proved inductively that $ba^{-r} = a^r b$. If n is even and $s = a^{n/2}$, then $s^{-1} = \left(a^{n/2}\right)^{-1} = a^{-(n/2)} = ea^{-(n/2)} = a^n a^{-(n/2)} = a^{n-n/2} = a^{n/2} = s$ since by the definition of dihedral group, $a^n = e$. Therefore, we proved that if $s = a^r b$ with $r \in \{0, 1, 2, \dots, n-1\}$ or $s = a^{n/2}$ if n is even, then s is self-inverse. On the contrary, let s be self-inverse. There are two cases of s based on the existence of b in s.

Case 1. If s contains b, then s can be written as $s = a^r b$ with $r \in \{0, 1, 2, \dots, n-1\}$. Observe that for arbitrary $r \in \{0, 1, 2, \dots, n-1\}$, $s^{-1} = (a^r b)^{-1} = b^{-1}(a^r)^{-1} = ba^{-r} = a^r b = s$. Therefore, if $s \in S$ is self-inverse and s contains b, then $s = a^r b$ with $r \in \{0, 1, 2, \dots, n-1\}$.

Case 2. If s does not contain b, then s can be written as $s = a^k$ with $k \in \{1, 2, 3, \dots, n-1\}$. Observe that $s^{-1} = (a^k)^{-1} = a^{-k} = ea^{-k} = a^n a^{-k} = a^{n-k}$. Then $s = s^{-1} \Longleftrightarrow a^k = a^{n-k} \Longleftrightarrow k = n - k \Longleftrightarrow k = \frac{n}{2}$ if n is even. Therefore, if $s \in S$ satisfies $s = s^{-1}$ and s does not contain b, then $s = a^{n/2}$ if n is even. ∎

Remark. From Observation 1, we can also conclude that $s \in S$ is not self-inverse if and only if $s = a^k$ with $k \in \{1, 2, 3, \dots, n-1\}$ and $k \neq \frac{n}{2}$.

3.3.1 Graph $Cay(D_{2n}, S)$ where $|s| = 1$

First, we give the simulation results of $\text{Cay}(D_{2n}, S)$ where $|S| = 1$. The sets S satisfying $|S| = 1$ are given in Lemma 2.

Lemma 2. Let $n \in \mathbb{Z}, n \geq 3$. Let D_{2n} be the dihedral group of order $2n$ and $e \in D_{2n}$ be the identity element. Let $S \subseteq D_{2n} - \{e\}$ which is inverse-closed. Then $|S| = 1$ if and only if S is one of the following sets:

1. $\{a^r b\}$ where $r \in \{0, 1, 2, \dots, n-1\}$, or
2. $\{a^{n/2}\}$ if n is even.

Proof. It is clear that if $S = \{a^r b\}$ with $r \in \{0, 1, 2, \dots, n-1\}$ or $S = \left\{a^{n/2}\right\}$ if n is even, then $|S| = 1$. On the contrary, let $|S| = 1$. Then we can write $S = \{s\}$ for some $s \in D_{2n} - \{e\}$. Since S is inverse-closed, then s must satisfy $s = s^{-1}$ and by Observation 1, we obtain two different sets S, that is, $S = \{a^r b\}$ with $r \in \{0, 1, 2, \dots, n-1\}$ or $S = \left\{a^{n/2}\right\}$ if n is even. ∎

The simulation results of $\text{Cay}(D_{2n}, S)$ for connection sets $S = \{a^r b\}$, $r \in \{0, 1, 2, \dots, n-1\}$ and $S = \{a^{n/2}\}$ if n is even are shown in Table 1 and Table 2, respectively.

Table 1. Simulation results of $\text{Cay}(D_{2n}, S)$ where $S = \{a^r b\}$, $r \in \{0, 1, 2, \dots, n-1\}$.

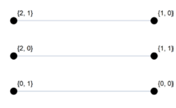
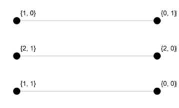
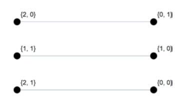
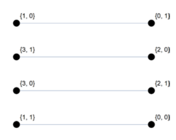
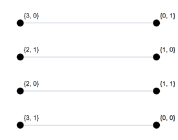
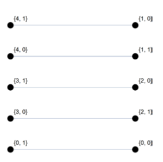
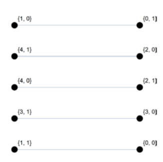
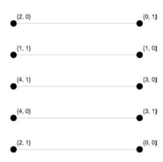
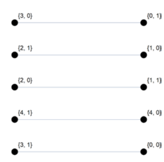
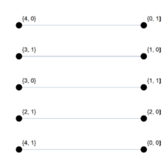
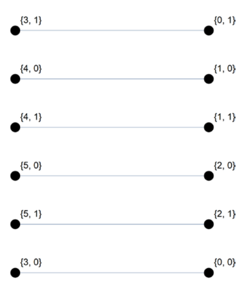
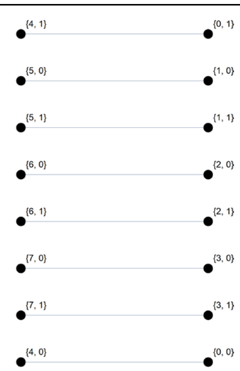
n	r				
	0	1	2	3	4
3					
4					
5					

Table 2. Simulation results of $\text{Cay}(D_{2n}, S)$ where $S = \{a^{n/2}\}$, n is even.

n	4	6	8
$\text{Cay}(D_{2n}, S)$			

3.3.2 Graph $Cay(D_{2n}, S)$ where $\lvert S \rvert = 2$

Next, we give the simulation results of $\text{Cay}(D_{2n}, S)$ where $|S| = 2$. The sets S satisfying $|S| = 2$ are given in Lemma 3.

Lemma 3. Let $n \in \mathbb{Z}$, $n \geq 3$. Let D_{2n} be the dihedral group of order $2n$ and $e \in D_{2n}$ be the identity element. Let $S \subseteq D_{2n} - \{e\}$ which is inverse-closed. Then $|S| = 2$ if and only if S is one of the following sets:

1. $\{a^k, a^{n-k}\}$ where $k \in \{1,2,3,\dots,n-1\}$, $k \neq \frac{n}{2}$, or
2. $\{a^r b, a^s b\}$ where $r, s \in \{0,1,2,\dots,n-1\}$, $r \neq s$, or
3. $\{a^r b, a^{n/2}\}$ if n is even and $r \in \{0,1,2,\dots,n-1\}$.

Proof. It is clear that if S is the set given in number 1, 2, or 3, then $|S| = 2$. On the contrary, let $|S| = 2$. Then we can write $S = \{s_1, s_2\}$ for some $s_1, s_2 \in D_{2n} - \{e\}$ and $s_1 \neq s_2$. There are some cases for s_1 and s_2 based on whether s_1 and s_2 are self-inverse or not.

Case 1. Suppose that s_1 and s_2 are self-inverse. By Observation 1 and the condition that $s_1 \neq s_2$, there are two subcases based on the existence of $a^{n/2}$ in S. **Subcase 1:** Suppose that $a^{n/2} \notin S$. Then $s_1 = a^r b$ and $s_2 = a^s b$ where $r, s \in \{0,1,2,\dots,n-1\}$, $r \neq s$. Therefore, $S = \{a^r b, a^s b\}$ and we obtain the set in number 2. **Subcase 2:** Suppose that $a^{n/2} \in S$. Without loss of generality, let $s_1 = a^r b$ where $r \in \{0,1,2,\dots,n-1\}$ and $s_2 = a^{n/2}$ if n is even. Then $S = \{a^r b, a^{n/2}\}$ and we obtain the set in number 3.

Case 2. Suppose s_1 and s_2 are not self-inverse. From the remark of Observation 1, we know that $s_1 = a^{k_1}$ and $s_2 = a^{k_2}$ for some $k_1, k_2 \in \{1,2,3,\dots,n-1\}$, $k_1 \neq k_2$, and $k_1, k_2 \neq \frac{n}{2}$. Then $S = \{a^{k_1}, a^{k_2}\}$. Since S is inverse-closed and a^{k_1}, a^{k_2} are not self-inverse, then a^{k_1} must be the inverse of a^{k_2} and vice versa. Thus $a^{k_2} = (a^{k_1})^{-1} = a^{-k_1} = ea^{-k_1} = a^n a^{-k_1} = a^{n-k_1}$. Therefore, by writing $k_1 = k$, we obtain $S = \{a^k, a^{n-k}\}$ with $k \in \{1,2,3,\dots,n-1\}$, $k \neq \frac{n}{2}$ which is the set in number 1.

Case 3. Without loss of generality, suppose s_1 is self-inverse and s_2 is not self-inverse. Since s_2 is not self-inverse and S is inverse-closed, then there exists an $s_3 \in S$ that differs from s_2 such that $s_3 = s_2^{-1}$. Since s_1 is self-inverse while s_3 is not self-inverse, then $s_3 \neq s_1$. It means that there exists an $s_3 \in S$ that differs from s_1 and s_2. Hence, $|S| = 3$ which contradicts the condition that $|S| = 2$. Therefore, this case is ineligible and ignored. ■

The simulation results of $\text{Cay}(D_{2n}, S)$ for connection sets S given in number 1, 2, and 3 in Lemma 3 are shown in Table 3, Table 4, and Table 5, respectively.

Table 3. Simulation results of $\text{Cay}(D_{2n}, S)$ where $S = \{a^k, a^{n-k}\}$, $k \in \{1,2,3,\dots,n-1\}$, $k \neq \frac{n}{2}$.

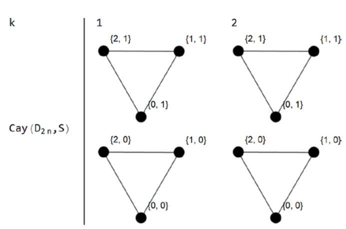
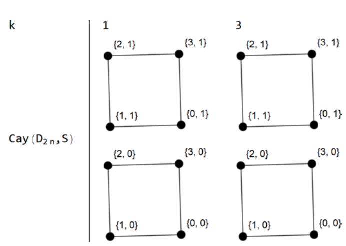
n	Simulation results
$n = 3$	
$n = 4$	

Table 3 (continued). Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^k, a^{n-k}\}$, $k \in \{1, 2, 3, \ldots, n-1\}$, $k \neq \frac{n}{2}$.

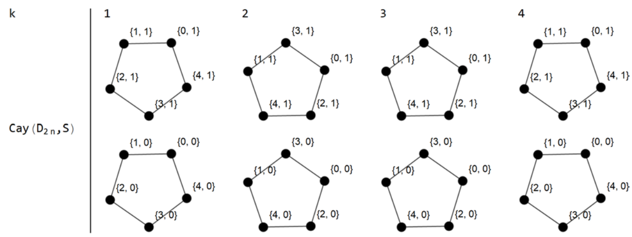
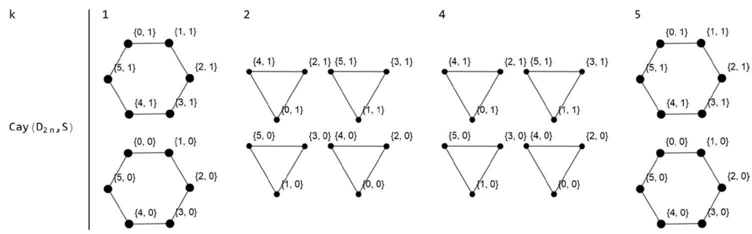
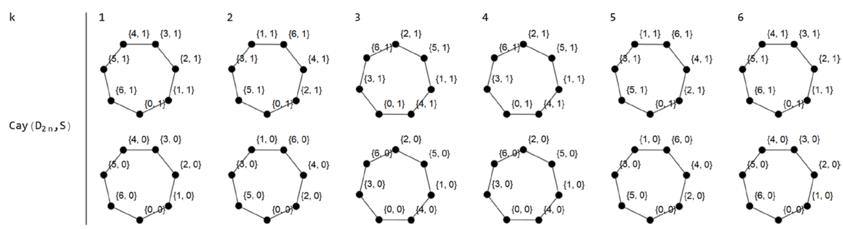
n	Simulation results
$n = 5$	
$n = 6$	
$n = 7$	

Table 4. Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^s b\}$ with $r, s \in \{0, 1, 2, \ldots, n-1\}$, $r \neq s$.

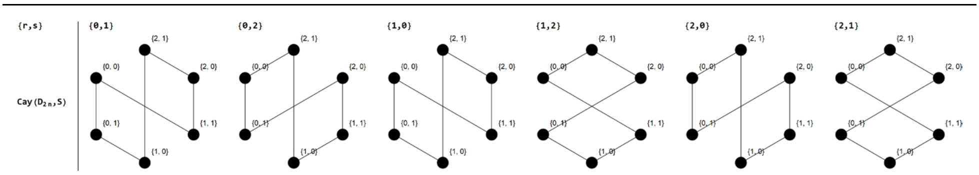
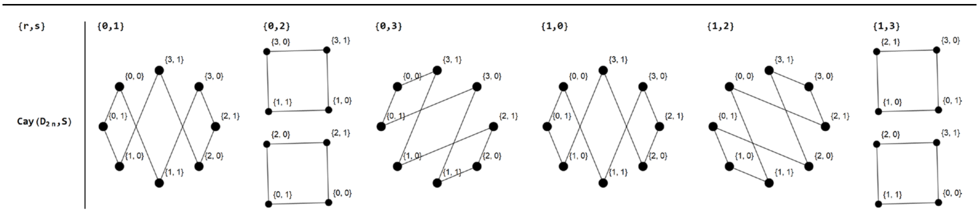
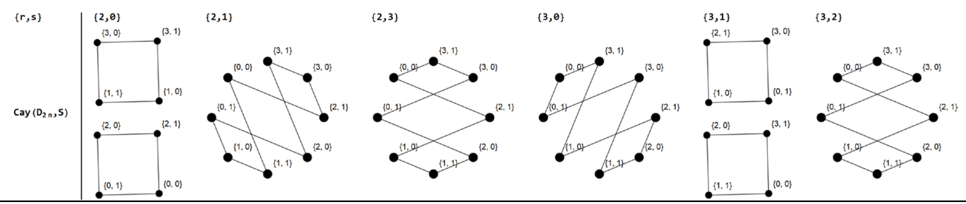
$n = 3$

$n = 4$


Table 4 (continued). Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^s b\}$ with $r, s \in \{0, 1, 2, \dots, n-1\}$, $r \neq s$.

$n = 4$



$n = 5$

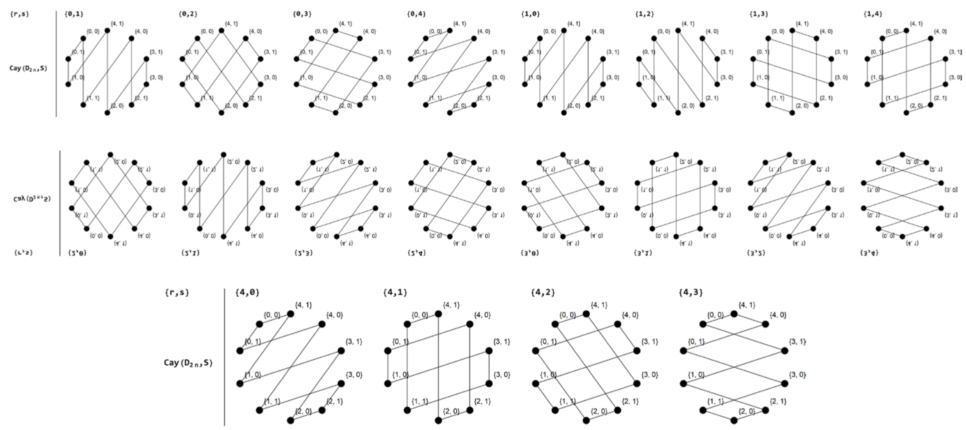
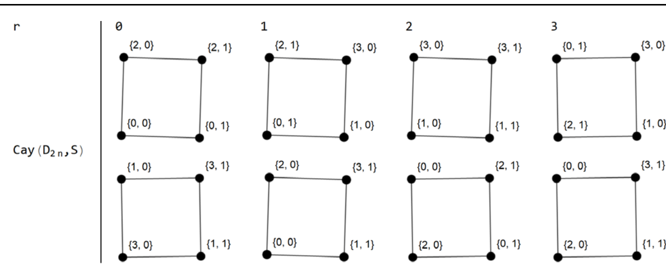


Table 5. Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^{n/2}\}$, $r \in \{0, 1, 2, \dots, n-1\}$, n is even.

$n = 4$



$n = 6$

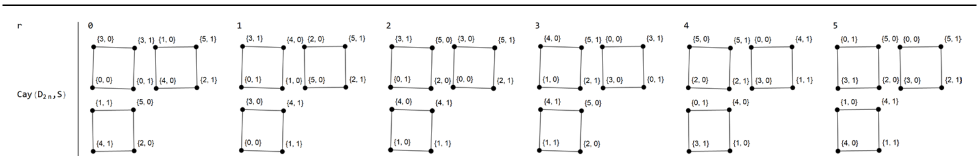
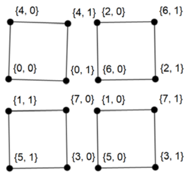


Table 5 (continued). Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^{n/2}\}$, $r \in \{0, 1, 2, \dots, n-1\}$, n is even.

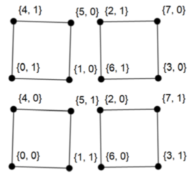
$n = 8$

r: 0

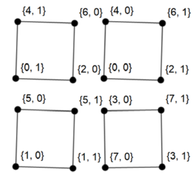
Cay (D_{2n}, S):



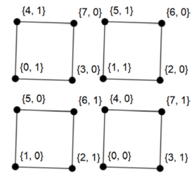
r: 1



r: 2

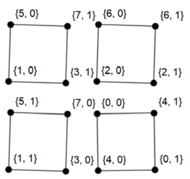


r: 3

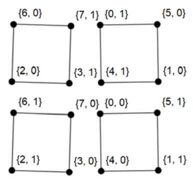


r: 4

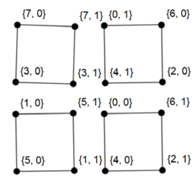
Cay (D_{2n}, S):



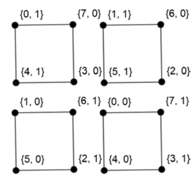
r: 5



r: 6



r: 7



3.3.3 Graph $Cay(D_{2n}, S)$ where $|S| = 3$

Finally, we give the simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $|S| = 3$. The sets S satisfying $|S| = 3$ are given in Lemma 4.

Lemma 4. Let $n \in \mathbb{Z}$, $n \geq 3$. Let D_{2n} be the dihedral group of order $2n$ and $e \in D_{2n}$ be the identity element. Let $S \subseteq D_{2n} - \{e\}$ which is inverse-closed. Then $|S| = 3$ if and only if S is one of the following sets:

1. $\{a^r b, a^s b, a^t b\}$ where $r, s, t \in \{0, 1, 2, \dots, n-1\}$ which are different from each other, or
2. $\{a^r b, a^s b, a^{n/2}\}$ where $r, s \in \{0, 1, 2, \dots, n-1\}$, $r \neq s$, n is even, or
3. $\{a^k, a^{n-k}, a^r b\}$ where $r \in \{0, 1, 2, \dots, n-1\}$, $k \in \{1, 2, 3, \dots, n-1\}$, $k \neq \frac{n}{2}$, or
4. $\{a^k, a^{n-k}, a^{n/2}\}$ where $k \in \{1, 2, 3, \dots, n-1\}$, $k \neq \frac{n}{2}$, n is even.

Proof. It is clear that if S is the set given in number 1, 2, 3, or 4, then $|S| = 3$. On the contrary, let $|S| = 3$. Then we can write $S = \{s_1, s_2, s_3\}$ for some $s_1, s_2, s_3 \in D_{2n} - \{e\}$ which are different from each other. There are some cases based on the number of elements of S that are self-inverse.

Case 1. Suppose that there are three self-inverse elements in S, that is s_1, s_2, and s_3. Then by Observation 1, there are two subcases based on the existence of $a^{n/2}$ in S. **Subcase 1:** Suppose $a^{n/2} \notin S$. Then $s_1 = a^r b$, $s_2 = a^s b$, and $s_3 = a^t b$ with $r, s, t \in \{0, 1, 2, \dots, n-1\}$ which are different from each other. Therefore, $S = \{a^r b, a^s b, a^t b\}$ and we obtain the set in number 1. **Subcase 2:** Suppose that $a^{n/2} \in S$. Without loss of generality, let $s_1 = a^r b$, $s_2 = a^s b$, and $s_3 = a^{n/2}$ with $r, s \in \{0, 1, 2, \dots, n-1\}$, $r \neq s$, and n is even. Then $S = \{a^r b, a^s b, a^{n/2}\}$ and we obtain the set in number 2.

Case 2. Suppose that there are two self-inverse elements in S. Without loss of generality, suppose that s_1 and s_2 are self-inverse and s_3 is not self-inverse. Then there exists an $s_4 \in S$ where $s_4 \neq s_3$ such that $s_4 = s_3^{-1}$. Since s_1 and s_2 are self-inverse while s_4 is not self-inverse, then $s_4 \neq s_1$ and $s_4 \neq s_2$. It means that there exists an $s_4 \in S$ which is different from s_1, s_2, and s_3. Hence $|S| = 4$ which contradicts the condition that $|S| = 3$. Therefore, this case is ineligible and ignored.

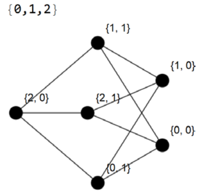
Case 3. Suppose that there is one self-inverse element in S. Without loss of generality, suppose s_3 is self-inverse while s_1 and s_2 are not self-inverse. Since s_1 and s_2 are not self-inverse and S is inverse-closed, then s_1 must be the inverse of s_2, that is $s_2 = s_1^{-1}$, and vice versa. From the remark of Observation 1, without loss of generality, let $s_1 = a^k$ and $s_2 = a^{n-k}$ with $k \in \{1, 2, 3, \dots, n-1\}$ and $k \neq \frac{n}{2}$. From Observation 1, there are two subcases as follows. **Subcase 1:** If $s_3 = a^r b$ where $r \in \{0, 1, 2, \dots, n-1\}$, then $S = \{a^k, a^{n-k}, a^r b\}$ and we obtain the set in number 3. **Subcase 2:** If n is even and $s_3 = a^{n/2}$, then $S = \{a^k, a^{n-k}, a^{n/2}\}$ and we obtain the set in number 4.

Case 4. Suppose that there is no self-inverse element in S so s_1, s_2, and s_3 are not self-inverse. Since S is inverse-closed, without loss of generality, let s_2 be the inverse of s_1, that is $s_2 = s_1^{-1}$. Since s_3 is not self-inverse, then there exists an $s_4 \in S$ where $s_4 \neq s_3$ such that $s_4 = s_3^{-1}$. Observe that if $s_4 = s_1$, then $s_3^{-1} = s_4 = s_1 = s_2^{-1} \Longrightarrow s_3 = s_2$ and if $s_4 = s_2$, then $s_3^{-1} = s_4 = s_2 = s_1^{-1} \Longrightarrow s_3 = s_1$. These results are contradicting the fact that s_1, s_2, and s_3 are different from each other. Then s_4 must be different from s_1 and s_2. It means that there exists an $s_4 \in S$ which is different from s_1, s_2, and s_3. Hence $|S| = 4$ which contradicts the condition that $|S| = 3$. Therefore, this case is ineligible and ignored. ■

The simulation results of $\mathrm{Cay}(D_{2n}, S)$ for connection sets S given in number 1, 2, 3, and 4 in Lemma 4 are shown in Tables 6, 7, 8, and 9, respectively. To simplify the presentation, every graph shown in Table 6 is $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^s b, a^t b\}$, $r, s, t \in \{0, 1, 2 \dots, n-1\}$, and $r < s < t$.

Table 6. Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^s b, a^t b\}$, $r, s, t \in \{0, 1, 2 \dots, n-1\}$ which are different from each other.

$n = 3$

{r,s,t}	{0,1,2}
Cay(D₂ₙ,S)	

$n = 4$

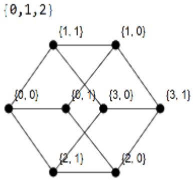
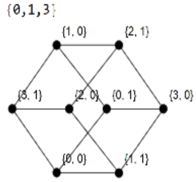
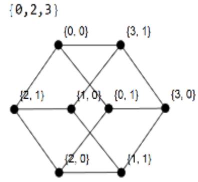
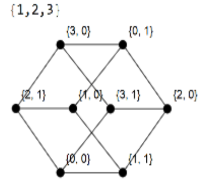
{r,s,t}	{0,1,2}	{0,1,3}	{0,2,3}	{1,2,3}
Cay(D₂ₙ,S)				

Table 6 (continued). Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^s b, a^t b\}$, $r, s, t \in \{0, 1, 2 \dots, n-1\}$ which are different from each other.

$n = 5$

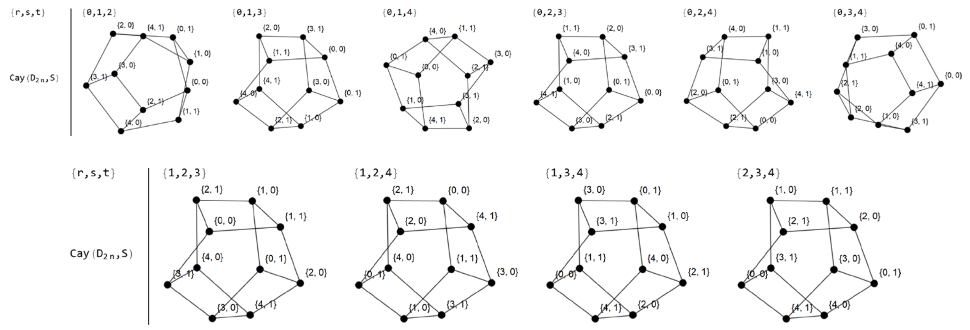
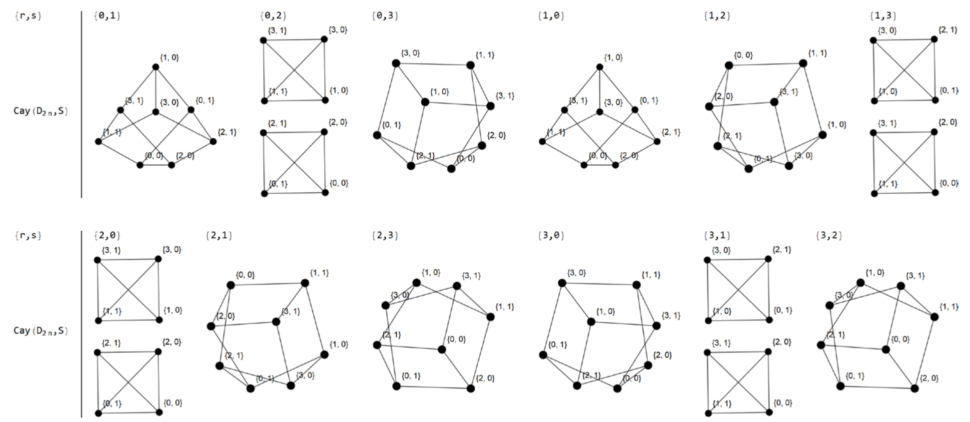


Table 7. Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^s b, a^{n/2}\}$, $r, s \in \{0, 1, 2, \dots, n-1\}$, $r \neq s$, n is even.

$n = 4$



$n = 6$

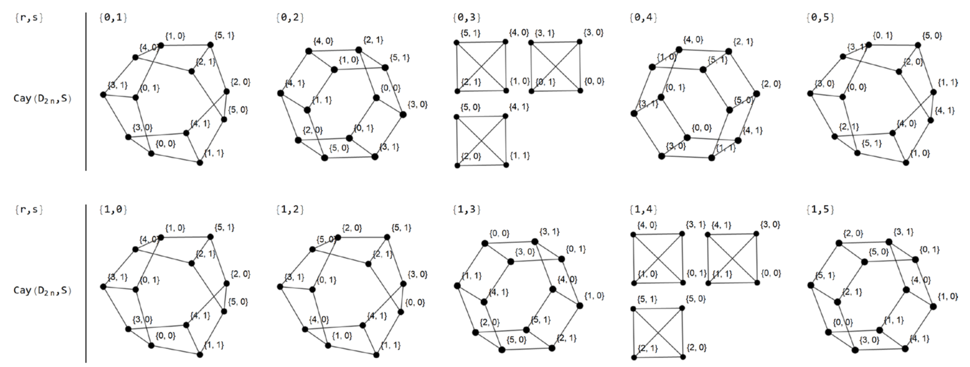


Table 7 (continued). Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^r b, a^s b, a^{n/2}\}$, $r, s \in \{0, 1, 2, \ldots, n-1\}$, $r \neq s$, n is even.

$n = 6$

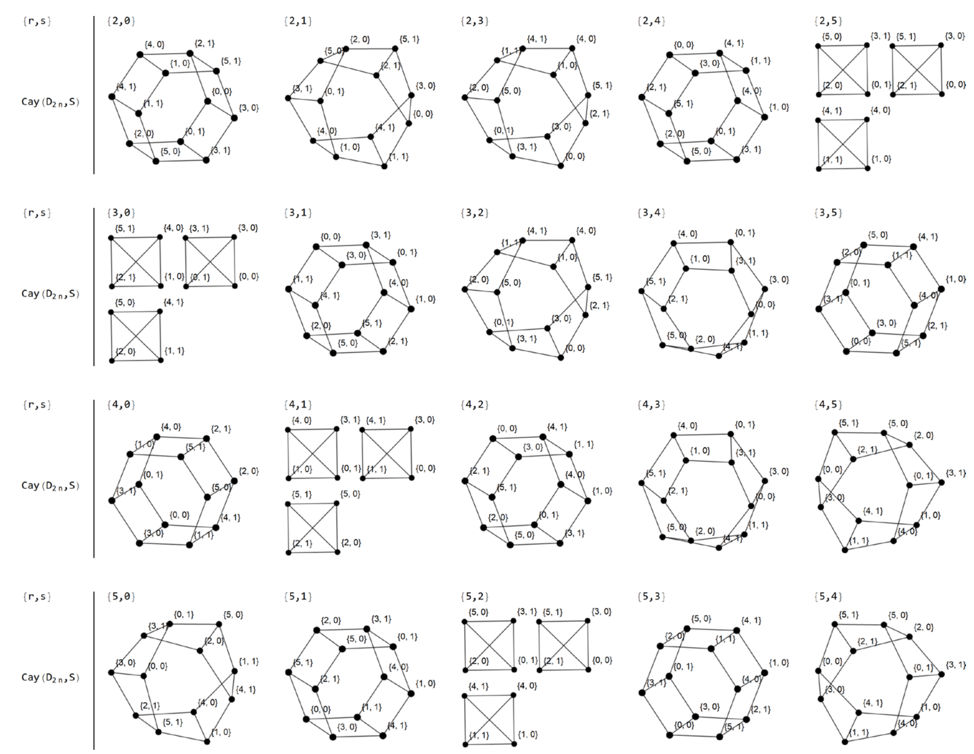
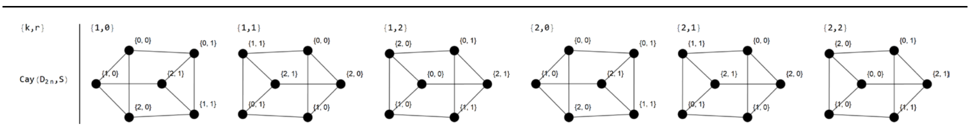


Table 8. Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^k, a^{n-k}, a^r b\}$, $r \in \{0, 1, 2, \ldots, n-1\}$, $k \in \{1, 2, \ldots, n-1\}$, $k \neq \frac{n}{2}$.

$n = 3$



$n = 4$

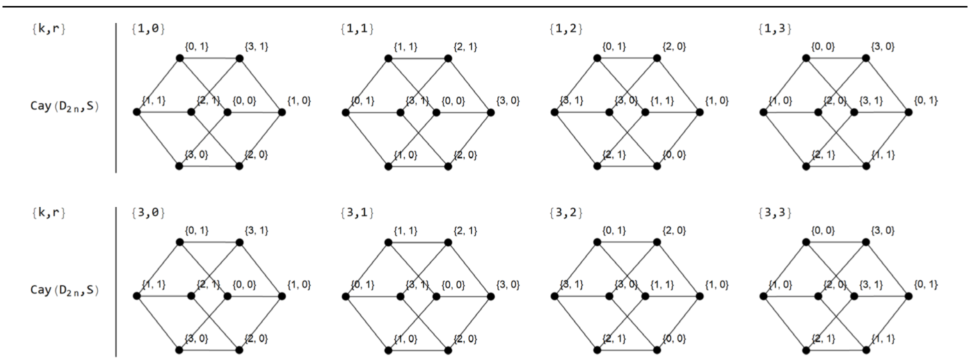


Table 8 (continued). Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^k, a^{n-k}, a^r b\}$, $r \in \{0, 1, 2, \dots, n-1\}$, $k \in \{1, 2, \dots, n-1\}$, $k \neq \frac{n}{2}$.

$n = 5$

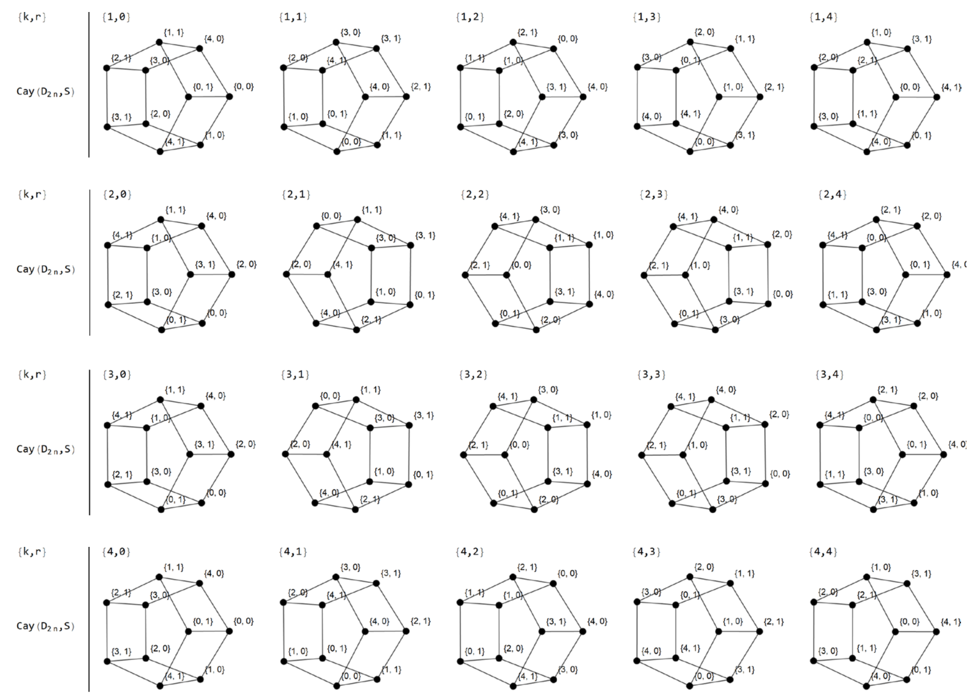
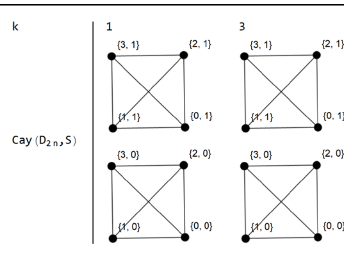


Table 9. Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^k, a^{n-k}, a^{n/2}\}$, $k \in \{1, 2, 3, \dots, n-1\}$, $k \neq \frac{n}{2}$, n is even.

$n = 4$



$n = 6$

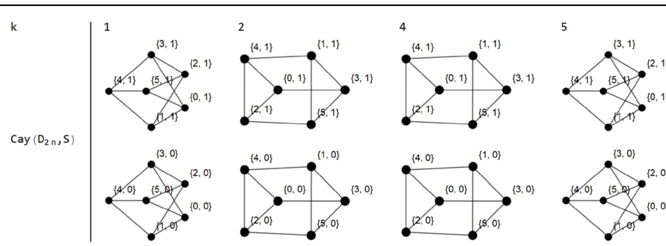
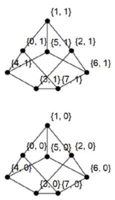
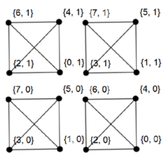
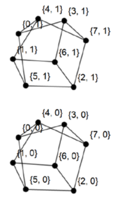
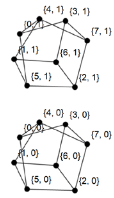
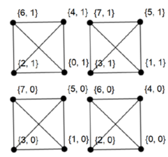
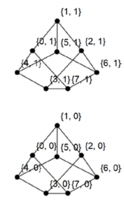


Table 9 (continued). Simulation results of $\mathrm{Cay}(D_{2n}, S)$ where $S = \{a^k, a^{n-k}, a^{n/2}\}$, $k \in \{1, 2, 3, \ldots, n-1\}$, $k \neq \frac{n}{2}$, n is even.

$n = 8$						
k	1	2	3	5	6	7
Cay (D_{2n}, S)						

4 Conclusion

In this paper, we conducted simulations of Cayley graphs of the dihedral group using Mathematica. By writing a program, we can construct Cayley graphs of dihedral group $\mathrm{Cay}(D_{2n}, S)$ for some connection sets S. Then we found all connection sets S where $|S| = 1, 2, 3$ that satisfy the conditions: $S \subseteq D_{2n} - \{e\}$ where e is the identity element of D_{2n} and S is inverse-closed. By using the program written in Figure 1 and using these connection sets S, we found the Cayley graphs of the dihedral group shown in the tables in this paper.

References

1. I. N. Herstein, *Abstract Algebra*, 3rd Ed. (Prentice Hall, Michigan, 1996).
2. W. A. Adkins and S. H. Weintraub, *Algebra: An Approach via Module Theory*, Vol. 136 (Springer Science & Business Media, New York, 1992).
3. G. Chartrand and P. Zhang, *A First Course in Graph Theory* (Courier Corporation, 2012).
4. X. Liu and S. Zhou, arXiv preprint arXiv:1809.09829 (2018).
5. M. Krebs and A. Shaheen, *Expander Families and Cayley Graphs: A Beginner's Guide* (Oxford University Press, Oxford, 2011).
6. S. Hoory, N. Linial, and A. Wigderson, Bull. Amer. Math. Soc. **43**, 439-561 (2006).
7. A. V. Kelarev, Australas. J. Comb. **25**, 73-78 (2002).
8. R. Akhtar et al., Electron. J. Comb. **16**, R117 (2009).
9. D. Kiani and M. M. H. Aghaei, Electron. J. Comb. **19**, P10 (2012).
10. D. Marušič, Discrete. Math. **46**, 49-54 (1983).
11. L. Babai, J. Comb. Theory Ser. B **27**, 180-189 (1979).