Alpha power inverse Weibull distribution: A new lifetime distribution with application to gastric cancer data

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Abstract. Lifetime data analysis has an essential role in various fields of science. In general, lifetime data have a skewed distribution pattern. The Weibull distribution is one of the most frequently used distributions for modeling lifetime data. However, the Weibull distribution is not suitable for modeling data with non-monotonous hazard functions, one of which is an upside-down bathtub shape. According to Sharma et al. (2015), the inverse version of several probability distributions can model the data with an upside-down bathtub shape, one of which is the inverse Weibull distribution. This paper explains the Alpha Power Inverse Weibull (APIW) distribution as a generalization version of the inverse Weibull distribution. This distribution is constructed by using the Alpha Power Transformation method. The modification is done by adding a shape parameter to the inverse Weibull distribution to increase flexibility. The characteristics of APIW distribution discussed include probability density function, distribution function, survival function, hazard function, and the r-th moment. The probability density function of APIW distribution is left-skewed and unimodal. In addition, the hazard function of APIW distribution has an upside-down bathtub shape. The parameters of this distribution are estimated by the maximum likelihood method. Finally, for illustration purposes, the data about the time until gastric cancer patients die are modelled with the inverse Weibull distribution, and the APIW distribution is given. The modeling result shows that the Alpha Power Inverse Weibull distribution is better at modeling the time until gastric cancer patients die data than the inverse Weibull distribution.

Keywords. Alpha Power Transformation, hazard function, inverse Weibull distribution, lifetime data, maximum likelihood method

1 Introduction

Lifetime data describes the time needed for an event to happen. Lifetime data has a positive real value and generally has a skewed distribution pattern. The analysis of lifetime data

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appears in various fields, such as medicine, biology, public health, epidemiology, engineering, economics, and demography [1]. Some examples of lifetime data are time until death for humans, time until failure for some electronic devices, and time until cancer patients recover.

In addition to the data distribution pattern, another thing that needs to be considered in modeling lifetime data is the hazard function [2]. According to [2], the shape of the hazard function can be monotone (increasing and decreasing) or non-monotone (bathtub and upside-down bathtub shape).

Weibull distribution is one of the most used distributions in modeling lifetime data. This distribution has a skewed pattern, and the hazard function can be constant or monotone (increasing or decreasing). However, several datasets, particularly in the engineering, medicine, and reliability fields, revealed a non-monotone shape for their hazard function [3]. For example, Langlands et al. [4] looked at data from 3878 instances of breast cancer in Edinburgh from 1954 to 1964 and found that mortality was low in the first year, then increased in the following years before gradually decreasing. It indicates that the data has an upside-down hazard function. Therefore, Weibull distribution is not suitable for modeling this kind of data. Surprisingly, the inverse version of some probability distributions shows the upside-down bathtub-shaped for their hazard rates and so can model the situation mentioned above [5], some of which are inverse Weibull, inverse Gaussian, and inverse Gamma distribution. Therefore, the inverse Weibull distribution can be an alternative to model data with an upside-down bathtub hazard function.

Many generalizations of the inverse Weibull distribution have been studied to extend the flexibility of the inverse Weibull distribution. For example, the generalized inverse Weibull distribution [6], gamma inverse Weibull distribution [7], and Marshall-Olkin extended inverse Weibull distribution [8]. In 2019, Basheer [9] proposed a new distribution called Alpha Power Inverse Weibull (APIW) distribution by using the Alpha Power Transformation method introduced by Mahdavi and Kundu [10]. This distribution is a generalization of inverse Weibull distribution. Therefore, it can better describe lifetime data than inverse Weibull distribution.

This paper aims to introduce a new distribution called Alpha Power Inverse Weibull distribution based on the method of Alpha Power Transformation with its mathematical properties. It includes the shapes of the density and hazard functions, as well as the moments, the moment generating function, and the quantile. The parameters of this distribution are estimated by using the maximum likelihood estimation (MLE) method. Finally, the application of the model to the real data is presented and compared to the fit attained by inverse Weibull distribution.

2 Methodology

2.1 Alpha Power Transformation Method

Mahdavi and Kundu [10] introduced a new method for generalizing distributions. The proposed method is called Alpha Power Transformation (APT). The procedure is done by adding a new parameter to obtain a family of distributions. If $F(x)$ is a cumulative density function of any distributions, the alpha power transformation of $F(x)$ is defined as follows:

$$F_{APT}(x) = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1}, & \alpha > 0, \alpha \neq 1 \\ F(x), & \alpha = 1 \end{cases}. \quad (1)$$

The corresponding probability density function of (3) is,

$$f_{APT}(x) = \begin{cases} \dfrac{\log \alpha}{\alpha - 1} f(x)\alpha^{F(x)}, & \alpha > 0, \alpha \neq 1 \\ f(x), & \alpha = 1 \end{cases}. \tag{2}$$

2.2 Formation of Alpha Power Inverse Weibull distribution

In this section, we will extend the inverse Weibull distribution by using the Alpha Power Transformation method. The pdf and cdf of inverse Weibull distribution are, respectively, as follows:

$$f_{IW}(x) = \begin{cases} \lambda\beta x^{-(\beta+1)} e^{-\lambda x^{-\beta}}, & x > 0, \lambda > 0, \beta > 0 \\ 0, & \text{others} \end{cases}. \tag{3}$$

and,

$$F_{IW}(x) = e^{-\lambda x^{-\beta}}, \qquad x > 0, \lambda > 0, \beta > 0. \tag{4}$$

Let X be the continuous random variable that has Alpha Power Inverse Weibull distribution with parameters $\alpha > 0, \lambda > 0, \beta > 0$. By substituting the pdf of inverse Weibull distribution given by Equation (3) to Equation (2) and the cdf of inverse Weibull distribution given by Equation (4) to Equation (1), we get the pdf and cdf of Alpha Power Inverse Weibull distribution, respectively, as follows:

$$f_{APIW}(x) = \begin{cases} \dfrac{\log \alpha}{\alpha - 1} \lambda\beta x^{-(\beta+1)} e^{-\lambda x^{-\beta}} \alpha^{e^{-\lambda x^{-\beta}}}, & x > 0, \alpha > 0, \alpha \neq 1, \lambda > 0, \beta > 0 \\ \lambda\beta x^{-(\beta+1)} e^{-\lambda x^{-\beta}}, & x > 0, \alpha = 1, \lambda > 0, \beta > 0 \\ 0, & \text{others} \end{cases}, \tag{5}$$

and,

$$F_{APIW}(x) = \begin{cases} \dfrac{\alpha^{e^{-\lambda x^{-\beta}}} - 1}{\alpha - 1}, & x > 0, \alpha > 0, \alpha \neq 1, \lambda > 0, \beta > 0 \\ e^{-\lambda x^{-\beta}}, & x > 0, \alpha = 1, \lambda > 0, \beta > 0 \\ 0, & x \leq 0 \end{cases}. \tag{6}$$

From Equation (6), we can write the survival function of Alpha Power Inverse Weibull distribution as follow,

$$S_{APIW}(x) = \begin{cases} \dfrac{\alpha}{\alpha - 1}\left(1 - \alpha^{e^{-\lambda x^{-\beta}} - 1}\right), & x > 0, \alpha > 0, \alpha \neq 1, \lambda > 0, \beta > 0 \\ 1 - e^{-\lambda x^{-\beta}}, & x > 0, \alpha = 1, \lambda > 0, \beta > 0 \end{cases}. \tag{7}$$

Figure 1 illustrates the graph of the pdf of Alpha Power Inverse Weibull distribution for different values of α, λ, and β. This figure shows that the PDF of Alpha Power Inverse Weibull distribution is left-skewed and unimodal.

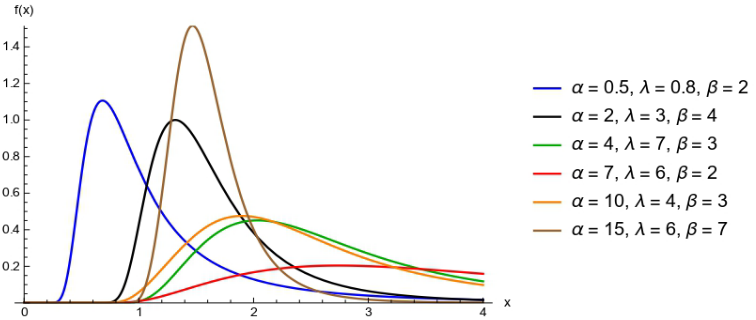

Fig. 1. Graph of the pdf of the Alpha Power Inverse Weibull distributions.

From Equation (5) and Equation (7), the hazard function of the Alpha Power Inverse Weibull distribution can be written as follows:

$$h_{APIW}(x) = \begin{cases} \log\alpha \cdot \lambda\beta x^{-(\beta+1)} e^{-\lambda x^{-\beta}} \dfrac{\alpha^{e^{-\lambda x^{-\beta}}-1}}{1-\alpha^{e^{-\lambda x^{-\beta}}-1}}, & x > 0, \alpha > 0, \alpha \neq 1, \lambda > 0, \beta > 0 \\ \dfrac{\lambda\beta x^{-(\beta+1)} e^{-\lambda x^{-\beta}}}{1-e^{-\lambda x^{-\beta}}}, & x > 0, \alpha = 1, \lambda > 0, \beta > 0 \end{cases}. \quad (8)$$

Figure 2 gives the graph of the hazard function of the Alpha Power Inverse Weibull distribution. From this figure, it is perceivable that the hazard function of Alpha Power Inverse Weibull distribution has an upside-down bathtub shape.

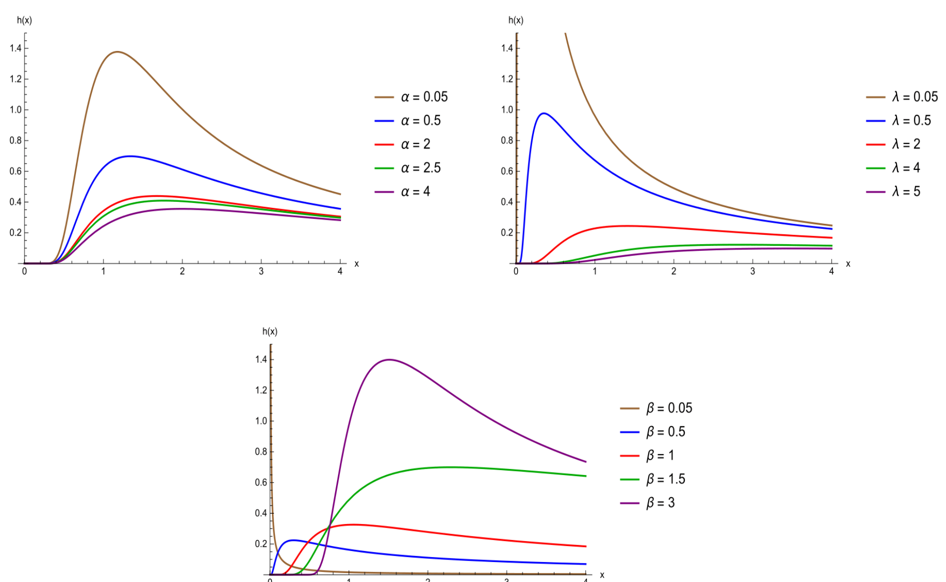

Fig. 2. Plot of the hazard rate of the Alpha Power Inverse Weibull distributions.

When $\alpha > 0, \alpha \neq 1$, by using the Taylor series expansion of the function $\alpha^{e^{-\lambda x^{-\beta}}}$, the pdf in Equation (5) can be written as follows:

$$f_{APIW}(x) = \frac{1}{\alpha - 1}\sum_{j=0}^{\infty}\frac{(\log\alpha)^{j+1}}{j!}\lambda\beta x^{-(\beta+1)}\, e^{-\lambda(j+1)x^{-\beta}}. \tag{9}$$

Using Equation (9), we get the r-th moment of the Alpha Power Inverse Weibull distribution, which is written as follows:

$$E(X^r) = \frac{\lambda^{\frac{r}{\beta}}}{\alpha - 1}\sum_{j=0}^{\infty}\frac{(\log\alpha)^{j+1}}{(j+1)!}(j+1)^{\frac{r}{\beta}} \cdot \Gamma\left(1 - \frac{r}{\beta}\right), \qquad r < \beta. \tag{10}$$

By substituting r=1 and r=2 in Equation (10), we can get the first and second moments of Alpha Power Inverse Weibull distribution, respectively, as follows:

$$E(X) = \frac{\lambda^{\frac{1}{\beta}}}{\alpha - 1}\sum_{j=0}^{\infty}\frac{(\log\alpha)^{j+1}}{(j+1)!}(j+1)^{\frac{1}{\beta}} \cdot \Gamma\left(1 - \frac{r}{\beta}\right), \tag{11}$$

and,

$$E(X^2) = \frac{\lambda^{\frac{2}{\beta}}}{\alpha - 1}\sum_{j=0}^{\infty}\frac{(\log\alpha)^{j+1}}{(j+1)!}(j+1)^{\frac{2}{\beta}} \cdot \Gamma\left(1 - \frac{2}{\beta}\right). \tag{12}$$

Based on Equation (11) and Equation (12), we can get variance of the Alpha Power Inverse Weibull distribution as follows:

$$\begin{aligned} Var(X) = \quad & \frac{\lambda^{\frac{2}{\beta}}}{\alpha - 1}\sum_{j=0}^{\infty}\frac{(\log\alpha)^{j+1}}{(j+1)!}(j+1)^{\frac{2}{\beta}} \cdot \Gamma\left(1 - \frac{2}{\beta}\right) \\ & - \left[\frac{\lambda^{\frac{r}{\beta}}}{\alpha - 1}\sum_{j=0}^{\infty}\frac{(\log\alpha)^{j+1}}{(j+1)!}(j+1)^{\frac{r}{\beta}} \cdot \Gamma\left(1 - \frac{r}{\beta}\right)\right]^2. \end{aligned} \tag{13}$$

The moment generating function (mgf) of Alpha Power Inverse Weibull distribution can be written as follows:

$$M(t) = \frac{1}{\alpha - 1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^k}{k!}\frac{(\log\alpha)^{j+1}}{(j+1)!}(j+1)^{\frac{k}{\beta}}\lambda^{\frac{k}{\beta}}\,\Gamma\left(1 - \frac{k}{\beta}\right), \qquad k < \beta\,. \tag{14}$$

The quantile of Alpha Power Inverse Weibull distribution is given by,

$$\pi_p = \left[\frac{1}{\lambda}\log\left(\frac{log\,\alpha}{log(p\alpha - p + 1)}\right)\right]^{-\frac{1}{\beta}}, \qquad 0 < p < 1. \tag{15}$$

2.3 Parameter estimation of Alpha Power Inverse Weibull distribution

We used the maximum likelihood estimation method for parameter estimation. Let $X_1, X_2, \dots, X_n$ be a random sample from Alpha Power Inverse Weibull distribution with parameters $\alpha > 0, \lambda > 0, \beta > 0$, then the likelihood function is given by,

$$L(\alpha, \lambda, \beta; x_1, \dots x_n) = \frac{(\log \alpha)^n}{(\alpha - 1)^n} \lambda^n \beta^n \left(\prod_{i=1}^{n} x_i^{-(\beta+1)} \right) e^{-\lambda \sum_{i=1}^{n} x_i^{-\beta}} \alpha^{\sum_{i=1}^{n} e^{-\lambda x_i^{-\beta}}}. \quad (16)$$

The logarithm of the likelihood function is then,

$$\begin{aligned} l(\alpha, \lambda, \beta; x_1, \dots x_n) = \; & \log(\log \alpha) - n \log(\alpha - 1) + n \log \lambda + n \log \beta \\ & -(\beta + 1) \sum_{i=1}^{n} \log(x_1) - \lambda \sum_{i=1}^{n} x_i^{-\beta} + \log \alpha \sum_{i=1}^{n} e^{-\lambda x_i^{-\beta}}. \end{aligned} \quad (17)$$

By taking the first partial derivatives of the log-likelihood function with respect to the three parameters, we get:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha \log \alpha} - \frac{n}{\alpha - 1} + \frac{1}{\alpha} \sum_{i=1}^{n} e^{-\lambda x_i^{-\beta}}, \quad (18)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i^{-\beta} - \log \alpha \sum_{i=1}^{n} x_i^{-\beta} e^{-\lambda x_i^{-\beta}}, \quad (19)$$

and,

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log x_i + \lambda \sum_{i=1}^{n} x_i^{-\beta} \log x_i + \lambda \log \alpha \sum_{i=1}^{n} x_i^{-\beta} \log x_i \, e^{-\lambda x_i^{-\beta}}. \quad (20)$$

The maximum likelihood estimates of α, λ, and β are obtained by solving the nonlinear equations $\frac{\partial l}{\partial \alpha} = 0$, $\frac{\partial l}{\partial \lambda} = 0$, and $\frac{\partial l}{\partial \beta} = 0$. Since these equations cannot be solved analytically, the estimates of the parameters α, λ, and β must therefore be found using numerical methods.

3 Results and discussion

3.1 Data illustration

In this section, we present an example to demonstrate the flexibility of this new distribution in modeling real data. The data in Table 1 represents the time until gastric cancer patients die (in days) as reported in Klein and Moeschberger [1]. Table 2 gives the descriptive statistics of the data. In Fig. 3, we see that the data is both left-skewed and unimodal. Furthermore, Fig. 4 shows the TTT plot of the data set. From Fig. 4, we can see that the TTT plot is concave-convex. It indicates that the shape of the hazard rate of the data is an upside-down bathtub. Therefore, we can consider Alpha Power Inverse Weibull distribution for modeling this data. For comparison purposes, we also model this data using inverse Weibull distribution. Table 3 displays the maximum likelihood estimates of the parameters. These values are obtained using RStudio version 4.0.2. Fig. 5 illustrates the fitted pdf and histogram of gastric cancer data, while Fig. 6 gives the empirical and fitted cdf. In Fig. 6, we can see that the cdf of Alpha Power Inverse Weibull distribution is close to the empirical cdf.

Table 1. Time until gastric cancer patients die (in days).

17	42	44	48	60	72	74	95	103	108
122	144	167	170	183	185	193	195	197	208
234	235	254	307	315	401	445	464	484	528
542	547	577	580	795	855	1366	1577	2060	

Table 2. Descriptive statistics of the gastric cancer data.

Mean	384.4359
Median	208
Variance	191643.3000
Skewness	2.2605
Kurtosis	5.1218

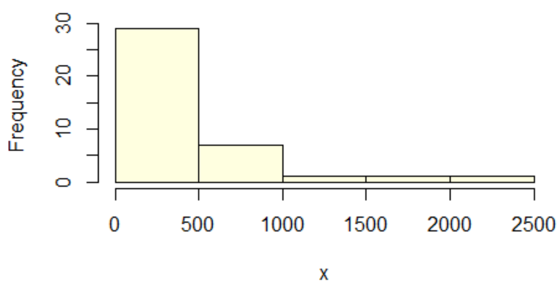

Fig. 3. Histogram of the gastric cancer data.

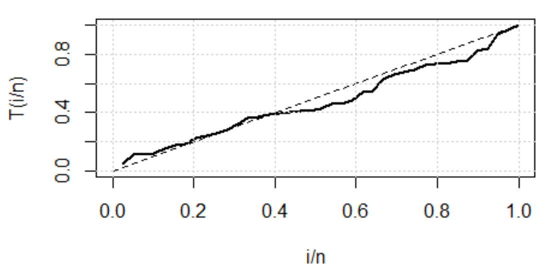

Fig. 4. TTT plot of the gastric cancer data.

Table 3. Maximum likelihood estimates of the parameters for data set.

Distribution	Estimates		
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$
Alpha Power Inverse Weibull	54.7689	93.8711	1.1636
Inverse Weibull	-	101.6940	0.9427

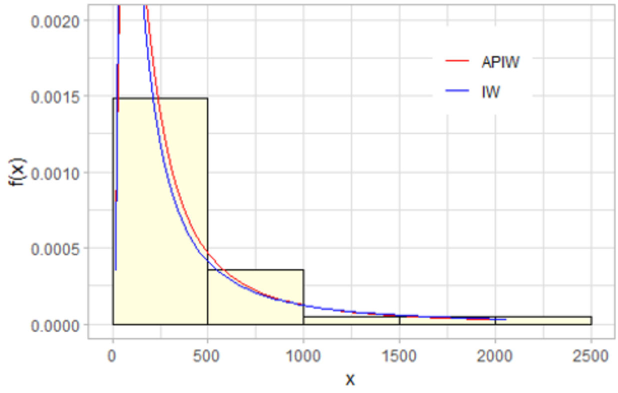

Fig. 5. The fitted pdf and histogram of gastric cancer data.

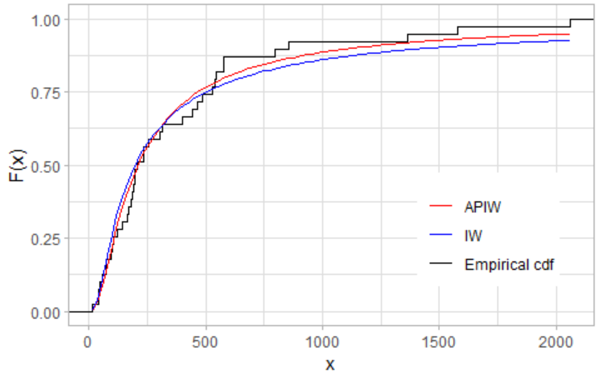

Fig. 6. The empirical cdf and fitted cdf of gastric cancer data.

To compare the fitted model Alpha Power Inverse Weibull distribution with the inverse Weibull distribution, we use the Kolmogorov-Smirnov test statistics, the Akaike Information Criterion (AIC) defined by $2k - 2\log(L)$, and the Bayesian Information Criterion (BIC) defined by $k\log(n) - 2\log(L)$, where n is the sample size, k is the number of parameters, and log (L) is the maximum value of log-likelihood function. The best-fit model would give the lowest value of Kolmogorov-Smirnov test statistic, AIC, and BIC. Table 4 provides information about the Kolmogorov-Smirnov test statistics, AIC, and BIC for the fitted Alpha Power Inverse Weibull and inverse Weibull distribution. By examining Table 4, we can conclude that the APIW model fits the data set better than the inverse Weibull distribution. Figure 7 tells that the hazard function of APIW distribution in this data set is an upside-down bathtub.

Table 4. Summary of the fitted models.

Distribution	Kolmogorov-Smirnov Test Statistics	AIC	BIC
Alpha Power Inverse Weibull	0.1028	547.8078	552.7885
542	547	577	580

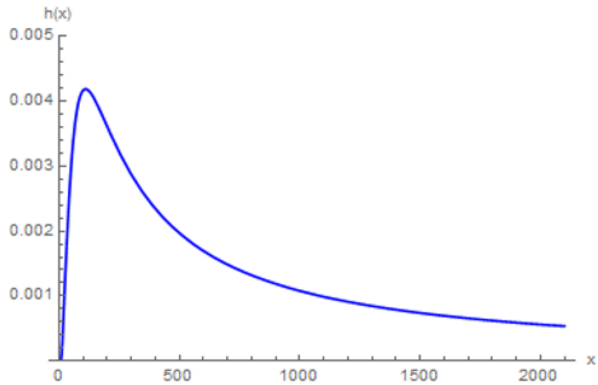

Fig. 7. The hazard function of Alpha Power Inverse Weibull distribution based on gastric cancer data.

4 Conclusion

In this paper, we introduced a new lifetime distribution, called Alpha Power Inverse Weibull (APIW) distribution, which extends the inverse Weibull distribution in the lifetime data analysis. The new model is obtained by implementing the Alpha Power Transformation method for generalizing distribution. The pdf of APIW distribution is unimodal and left-skewed. Moreover, the shape of the hazard function of this distribution is an upside-down bathtub, which generally appears in lifetime data. We also discuss some statistical properties of this distribution, such as the cdf, the survival function, the moment, moment generating function, and the quantile function in explicit forms. The estimation of parameters is approached by applying the maximum likelihood method. The simulation results on gastric cancer data show that the Alpha Power Inverse Weibull distribution can yield a better fit when modeling the gastric cancer data than the inverse Weibull distribution.

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