A preliminary study on linear perturbation for a non-minimal derivative coupling scalar-tensor theory

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Abstract. The Cisterna-Delsate-Rinaldi (CDR) model is a variant of scalar-tensor theory that modify gravity by including a term of non-minimal derivative coupling. This model gives interesting aspects in the properties of compact objects, specifically neutron stars. By adjusting one of its parameters, the maximum possible mass of neutron stars can be increased. The authors of the model had also did a perturbation analysis using odd-parity perturbation and following that they also did analysis on the slowly-rotating neutron stars. In this paper, we report our ongoing research on the linear perturbation for the Cisterna model to see its dynamical properties. More precisely, we work on the polar perturbation that affected both the metric and the scalar field, which is different from the axial perturbation used in the slow rotation case. We use higher-dimensional spacetimes to see if the obtained equations will be dimensionally dependent. To simplify calculations for this metric form, we use tetrad method. Currently, we have not succeeded in obtaining the equations of motions in the form of Regge-Wheeler-Zerilli wave equation. The reason is the metric functions cannot be easily decoupled and we find no second derivatives with respect to both time t and radius r in the equations of motion. Only the scalar field can give a wave equation. Further investigation is undergoing.

Keywords. Linear perturbation, non-minimal derivative coupling

1 Introduction

The CDR model is a variant of scalar-tensor model which actually came from the Fab Four model proposed by Charmousis et al. [1] (and independently by Deffayet et al. [2]). The model, which consists of no scalar potential term and four Lagrangian terms named after the members in The Beatles band, was originally proposed to study cosmology by establishing a unique action that allow a consistent self-tuning mechanism on FLRW backgrounds. The model is an attempt to solve the cosmological constant problem by employing the Horndeski's scalar tensor theory [3].

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Charmousis and Iosifidis had discussed the possibility of a black-hole solution with a non-trivial scalar hair from considering only the John term from the Fab four model and the usual minimal derivative coupling term [4]. This model turns out gives interesting solutions, such as, a stealth Schwarzschild black-hole and a partially self-tuned de-Sitter-Schwarzschild black-hole in a paper by Babichev and Charmousis [5], illustrating how such models can violate the no-hair theorem, with $\Phi = Qt + F(r)$ as the form of the non-trivial scalar field. (This form may not be a general solution, however, as Rinaldi [6] had found a black-hole solution with different from for the scalar field.) Here Q may seem like the scalar's charge, however, it is not true. This constant Q is used to accompany time variable t to emphasize that the scalar field Φ, in the static spacetimes, cannot be separated as a product of two functions with only one input, e.g., $\Phi(t, r) = T(t)R(r)$.

In the CDR model, an interesting technique is also used, i.e., the radial component of the scalar current be set to zero $J^r = 0$. This option is possible because the CDR model satisfies the symmetry of the Galileon model provided that the model is in a spherically symmetric space-time, as shown by Hui and Nicolis [7]. From symmetry only, Hui and Nicolis shown that the invariance under time-translations and rotation implies zero time and angular components, $J^t = 0 = J^\theta = J^\varphi$. From considering regularity on the boundary and also regularity on all range of radial variable r, Hui and Nicolis also showed that $J^r = 0$. Babichev and Charmousis [5] had shown that two different regularity conditions are satisfied by ansatz $\Phi = Qt + F(r)$. Those conditions are (1) the scalar field should be shift invariant $\Phi \to \Phi + \text{const.}$ so as to admit a no-hair theorem and (2) the scalar current $J^a J_a$ does not explode at the horizon. Moreover, the Lagrangian has no potential term, therefore the scalar field's contribution to the Lagrangian is only by its first derivative form. The conditions $J^a = 0 (a = t, r, \theta, \varphi)$ set constraints to the modified Einstein field equation, making some simplifications to the equations of motions from the modified Einstein field equation possible.

Cisterna *et al.* had used their model to study static compact object in [8]. Following the previous paper also in the same year, Cisterna *et al.* also study the axial perturbation of the model in [9]. In the following year, Cisterna *et al.* had studied the model in cosmological setting and investigate slowly-rotating neutron stars in the model [10]. Interestingly, their model does not approach general relativity solution by setting $\Lambda = \alpha = \eta = 0$. For more discussions about the CDR model, see a review paper by Olmo, Rubiera-Garcia, and Wojnar [11].

In this work, we attempt to continue the study of their metric perturbation, but in another form called the polar perturbation, which as far as we know from the literatures, had not yet been studied. According to Chandrasekhar [12], metric perturbation consists of two types of perturbation that are decoupled from each other up to linear terms: axial and polar perturbations. The axial perturbation is usually used to calculate the impact of a slowly rotating star [13]. This perturbation only perturbs the spacetime while the matter is left untouched. The latter affects both the spacetime and the matter and is related to another method to obtain the tidal deformation for compact stars (for instance, see [14]).

In the following section, we briefly discuss the model. We then follow the discussion with a subsection where we set our metric perturbation ansatz, another subsection about how we do our calculation, and some subsubsections showing our results. Then, we continue the implication of our results in a discussion section and finish it with a conclusions section.

2 CDR scalar-tensor gravity model

Action include both minimal and non-minimal derivative coupling to a real scalar field is given by,

$$S = \int \sqrt{-g} d^4x \left[\kappa(R - 2\Lambda) - \frac{1}{2}(\alpha g^{ab} - \eta G^{ab})\nabla_a\Phi\nabla_b\Phi \right] + S_m,$$

with $\kappa = 1/(16\pi G)$. Here we use the appropriate units such that $c = 1$. Variational principle used here to obtain the following equations of motion:

$$\begin{aligned} \nabla_a T^{ab} &= 0, \\ \nabla_a J^a &= 0, \\ G_{ab} + \Lambda g_{ab} - H_{ab} &= \frac{1}{2\kappa} T_{ab}. \end{aligned}$$

The scalar current is given as,

$$J^a = (\alpha g^{ab} - \eta G^{ab})\nabla_b\Phi,$$

and scalar contributions in the modified Einstein field equation (EFE) is,

$$H_{ab} = \sum_{n=1}^{2} \frac{\alpha}{2\kappa} H_{ab}^{(n)} + \sum_{n=3}^{11} \frac{\eta}{2\kappa} H_{ab}^{(n)}.$$

This consists of two contributions. Terms with coupling constant α and η denotes terms from minimal and non-minimal derivative coupling. The former terms are,

$$H_{ab}^{(1)} = \nabla_a\Phi\nabla_b\Phi,\ H_{ab}^{(2)} = -\frac{1}{2} g_{ab}\nabla^c\Phi\nabla_c\Phi,$$

and the latter terms are

$$\begin{aligned} &H_{ab}^{(3)} = \frac{1}{2} R\nabla_a\Phi\nabla_b\Phi,\ H_{ab}^{(4)} = -\nabla^c\Phi(\nabla_a\Phi R_{bc} + \nabla_b\Phi R_{ac}),\ H_{ab}^{(5)} = -\nabla^c\Phi\nabla^d\Phi R_{cadb}, \\ &H_{ab}^{(6)} = -\nabla_a\nabla^c\Phi\nabla_b\nabla_c\Phi,\ H_{ab}^{(7)} = \frac{1}{2} g_{ab}\nabla^c\nabla^d\Phi\nabla_c\nabla_d\Phi,\ H_{ab}^{(8)} = -\frac{1}{2} g_{ab}\left(\nabla_c\nabla^c\Phi\right)^2, \\ &H_{ab}^{(9)} = \nabla_c\nabla^c\Phi\nabla_a\nabla_b\Phi,\ H_{ab}^{(10)} = \frac{1}{2} g_{ab}\nabla^c\Phi\nabla_c\Phi G_{ab},\ H_{ab}^{(11)} = g_{ab}R_{cd}\nabla^c\Phi\nabla^d\Phi. \end{aligned}$$

2.1 Ansatz for linear perturbation

In our work, we follow Cisterna's assumptions that there is no minimal coupling contribution, no cosmological constant $\Lambda = \alpha = 0$ and we consider only the exterior solution $T_{ab} = 0$. Cisterna and colleageus had also obtained the full solution for black-hole case with nonzero α. However, in this work, we shall use a higher-dimensional space-time setting with the extra dimensions contained in $(D-2)$-sphere.

$$ds^2 = -f(t,r)dt^2 + g(t,r)dr^2 + h(t,r)r^2 d\Omega^2_{(D-2)}.$$

The spacetime is set to have arbitrary dimension D, with higher dimensional axes contained in $(D-2)$-sphere, in the hope that we can see if the result may be dimensionally dependent. Rather than the Christoffel symbol definitions for the Riemann tensor, we use the tetrad method (explained below) to obtain the non-zero Riemann tensor components. This method is less tedious to use for higher dimensional spacetimes.

In the static case, since the exterior solution should coincide with general relativity theory, we can use Tangherlini solution [15]:

$$f_0(r) = \frac{1}{g_0(r)} = 1 - \frac{r_h}{r^{D-3}}.$$

We then expand this metric only up to a linear term with δ is a bookkeeping parturbation parameter:

$$\begin{aligned} f(t,r) =& f_0(r) + \delta\Psi_f(t,r), \\ g(t,r) =& g_0(r) + \delta\Psi_g(t,r), \\ h(t,r) =& 1, \end{aligned}$$

where in the last line we assume no change in the radius of the $(D-2)$-sphere $\sqrt{h}r$ and the scalar field also expanded similarly,

$$\Phi(t,r) = Qt + F(r) + \delta\Psi_\Phi(t,r).$$

For now, we only want to see the pattern whether these ansatz may or may not produce linear differential equations whose form resembles that of Regge-Wheeler-Zerilli equation [12]:

$$A\left(B\Psi'\right)' - C\ddot{\Psi} + D\Psi = 0.$$

It should be mentioned that Chandrasekhar had used a sufficiently stationary space-time metric more general than this but his is strictly 4 dimensional. The motive why we want to obtain this form of equation is because we want to know the behavior of the gravitational wave. If the wave is decaying like a damped oscillator, then the model produces stable spacetimes. If the system is unstable, the wave may have an increase in amplitude without stopping.

2.2 Tetrad method

In this work, we follow Chadrasekhar's tetrad method as our calculation method. Tetrad method starts from non-orthonormal basis $\underline{e}^{\hat{a}}$ which is a 1-form with a different indices with hat to denote a tetrad indices:

$$\underline{e}^{\hat{a}} = e^{\hat{a}}{}_a dx^a; \quad \underline{\tilde{e}}^{\hat{i}} = \tilde{e}^{\hat{i}}{}_i d\theta^i.$$

These came from diagonalizing the metric:

$$\begin{aligned} ds^2 &= g_{ab}dx^a dx^b = \eta_{\hat{a}\hat{b}}\underline{e}^{\hat{a}} \otimes \underline{e}^{\hat{b}}, \\ d\Omega^2_{(D-2)} &= \tilde{g}_{ij}d\theta^i d\theta^j = \delta_{\hat{i}\hat{j}}\underline{\tilde{e}}^{\hat{i}} \otimes \underline{\tilde{e}}^{\hat{j}}. \end{aligned}$$

Here $\otimes$ denotes symmetric tensor product, $\eta_{\hat{a}\hat{b}} = diag(-1,1,1,1,...)$ denotes Minkowski metric, and $\delta_{\hat{i}\hat{j}}$ is the Kronecker delta. The spacetime indices a,b and tetrad indices $\hat{a},\hat{b}$ are set as follows:

$$\begin{aligned} a,b &= t,r,i,j; \quad i,j = \theta_2,\theta_3,...,\theta_{D-1}; \\ \hat{a},\hat{b} &= 0,1,\hat{i},\hat{j}; \quad \hat{i},\hat{j} = 2,3,...,D-1. \end{aligned}$$

These hats are just to differentiate between tetrad and space-time indices, but the entries are basically the same. The indices i, j and $\hat{\imath}$, $\hat{\jmath}$ are for the $(D-2)$-space. The $(D-2)$-space has the properties of maximally symmetric spaces:

$$\tilde{R}_{abcd} = k(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc}).$$

For sphere, flat, and hyperbolic spaces, $k = 1$, $k = 0$, and $k = -1$ respectively. In this work, we choose $(D-2)$-spheres as the $(D-2)$-spaces.

We then employ zero torsion $\underline{T}^{\hat{a}}$ to get the spin-connection 1-form $\underline{\omega}^{\hat{a}}{}_{\hat{b}}$:

$$0 = \underline{T}^{\hat{a}} = d\underline{e}^{\hat{a}} + \underline{\omega}^{\hat{a}}{}_{\hat{b}} \wedge \underline{e}^{\hat{b}}.$$

The basis of any n-form $(n > 1)$ behave antisymmetrically:

$$dx^a \wedge dx^b = -dx^b \wedge dx^a = \frac{1}{2}\left(dx^a \otimes dx^b - dx^b \otimes dx^a\right).$$

The antisymmetric behavior is also applied to the indices of $\underline{\omega}_{\hat{a}\hat{b}}$:

$$\underline{\omega}_{\hat{a}\hat{b}} = -\underline{\omega}_{\hat{b}\hat{a}}.$$

To lower or rise these indices, one use the Minkowski metric $\eta_{\hat{a}\hat{b}} = \eta^{\hat{a}\hat{b}} = diag(-1, 1, 1, 1, \ldots)$. All the n-forms (except exterior derivative d) are denoted by an underline. After obtaining the spin-connection, we use

$$\underline{R}^{\hat{a}}{}_{\hat{b}} = d\underline{\omega}^{\hat{a}}{}_{\hat{b}} + \underline{\omega}^{\hat{a}}{}_{\hat{c}} \wedge \underline{\omega}^{\hat{c}}{}_{\hat{b}}$$

to get the curvature 2-form $\underline{R}^{\hat{a}}{}_{\hat{b}}$, whose entries is just the Riemann tensor $R^{\hat{a}}{}_{\hat{b}cd}$:

$$\underline{R}^{\hat{a}}{}_{\hat{b}} = \frac{1}{2} R^{\hat{a}}{}_{\hat{b}cd}\, dx^c \wedge dx^d.$$

Indices of $R_{\hat{a}\hat{b}}$ is also antisymmetric as $\underline{\omega}_{\hat{a}\hat{b}}$. Lastly, we convert all the tetrad indices back into space-time indices using both $e^{\hat{a}}{}_b$ (components of the non-orthonormal basis $\underline{e}^{\hat{a}}$) and $e_{\hat{a}}{}^c$ (related to $e^{\hat{a}}{}_b$ by $e^{\hat{a}}{}_b e_{\hat{a}}{}^c = \delta^c_b$) so that we can get $R^a{}_{bcd}$:

$$R^a{}_{bcd} = R^{\hat{a}}{}_{\hat{b}cd}\, e_{\hat{a}}{}^a e^{\hat{b}}{}_b.$$

We shall show only the results in the following subsubsections.

2.2.1 Einstein tensor components

Below are the only non-zero components from the Einstein tensor. For a function $f(t, r)$, we denote its partial derivative with respect to time and radius as $\partial f/\partial t = \dot{f}$ and $\partial f/\partial r = f'$, respectively.

$$G_{tt} = \frac{(D-2)f_0}{r}\left\{\frac{g_0'}{2g_0^2} - \frac{(D-3)}{2r}\left(\frac{1}{g_0} - 1\right)\right\}$$
$$+ \delta\left[\frac{(D-2)f_0}{r}\left\{\frac{\Psi_g'}{2g_0^2} - \frac{g_0'\Psi_g}{g_0^3} + \frac{(D-3)}{2r}\frac{\Psi_g}{g_0^2} + \frac{\Psi_f}{f_0}\left\{\frac{g_0'}{2g_0^2} - \frac{(D-3)}{2r}\left(\frac{1}{g_0} - 1\right)\right\}\right\}\right],$$

$$
\begin{aligned}
G_{rr} =& \frac{(D-2)g_0}{r}\left\{\frac{f_0'}{2f_0g_0}+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\\
&+\delta\left[\frac{(D-2)g_0}{r}\left\{\frac{f_0'}{2f_0g_0}\left(\frac{\Psi_f'}{f_0'}-\frac{\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right)-\frac{(D-3)}{2r}\frac{\Psi_g}{g_0^2}\right.\right.\\
&\left.\left.+\frac{\Psi_g}{g_0}\left\{\frac{f_0'}{2f_0g_0}+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\right\}\right],
\end{aligned}
$$

$$
G_{tr} = \delta\frac{(D-2)\dot{\Psi}_g}{2rg_0},
$$

$$
\begin{aligned}
G_{ij} =& g_{ij}\left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}}+\frac{(D-3)}{2rg_0}\left(\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right)+\left(\frac{1}{g_0}-1\right)\frac{(D-3)(D-4)}{2r^2}\right\}\\
&+\delta\left[g_{ij}\left\{-\frac{\ddot{\Psi}_g}{2f_0g_0}+\frac{f_0''}{2f_0g_0}\left(\frac{\Psi_f''}{f_0''}-\frac{\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right)-\frac{f_0'^2}{4f_0^2g_0}\left(-\frac{2\Psi_f'}{f_0'}+\frac{2\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right)\right.\right.\\
&\left.\left.+\frac{f_o'g_0'}{4f_0g_0^2}\left(-\frac{\Psi_f'}{f_0'}-\frac{\Psi_g'}{g_0'}+\frac{\Psi_f}{f_0}+\frac{2\Psi_g}{g_0}\right)+\frac{(D-3)g_0'}{2rg_0^2}\left(-\frac{\Psi_g'}{g_0'}+\frac{2\Psi_g}{g_0}\right)+\frac{(D-3)(D-4)}{2r^2}\frac{\Psi_g}{g_0^2}\right\}\right].
\end{aligned}
$$

2.2.2 Scalar current J^a

The nonzero scalar current components are shown below:

$$
\begin{aligned}
J^r =& -\eta\frac{F'}{g_0}\left[\frac{(D-2)\sqrt{f_0}'}{\sqrt{f_0}rg_0}+\left(\frac{1}{g_0}-1\right)\frac{(D-3)(D-2)}{2r^2}\right]\\
&+\delta\left\{-\eta\frac{F'}{g_0}\left[\frac{(D-2)\sqrt{f_0}'}{\sqrt{f_0}rg_0}\left(\frac{\Psi_f'}{f_0'}-\frac{\Psi_g}{g_0}-\frac{\Psi_f}{f_0}\right)+\frac{\Psi_g}{g_0^2}\frac{(D-3)(D-2)}{2r^2}\right]\right.\\
&+\left[\frac{(D-2)\sqrt{f_0}'}{\sqrt{f_0}rg_0}+\left(\frac{1}{g_0}-1\right)\frac{(D-3)(D-2)}{2r^2}\right]\left[-\eta\frac{\Psi_\Phi'}{g_0}+\eta\frac{F'\Psi_g}{g_0^2}\right]\\
&\left.+\frac{\eta(D-2)Q\dot{\Psi}_g}{2f_0g_0^2}\right\},
\end{aligned}
$$

$$
\begin{aligned}
J^t =& \frac{\eta Q}{f_0}\frac{(D-2)}{r}\left\{\frac{1}{\sqrt{g_0}}\left(\frac{1}{\sqrt{g_0}}\right)'+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\\
&+\delta\left[\frac{\eta Q}{f_0}\frac{(D-2)}{r}\left\{\frac{1}{\sqrt{g_0}}\left(\frac{1}{\sqrt{g_0}}\right)'\left(\frac{\Psi_g'}{g_0'}-\frac{2\Psi_g}{g_0}\right)+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\frac{\Psi_g}{g_0(g_0-1)}\right\}\right.\\
&\left.+\frac{\eta Q}{f_0}\frac{(D-2)}{r}\left\{\frac{1}{\sqrt{g_0}}\left(\frac{1}{\sqrt{g_0}}\right)'+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\left(-\frac{\Psi_f}{f_0}+\frac{\dot{\Psi}_\Phi}{Q}\right)\right].
\end{aligned}
$$

2.2.3 Scalar current conservation $\nabla_a J^a = 0$

From $O(1)$ in $\nabla_a J^a = 0$ we obtain

$$
\left\{-\eta\frac{F'}{g_0}\frac{(D-2)}{r}\left[\frac{\sqrt{f_0}'}{\sqrt{f_0}g_0}+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right]\right\}' = 0.
$$

This is compatible with the choice $J^r = 0$ for the static case, which implies $G_{rr}(\delta^0) = 0$.

From $O(\delta)$ in $\nabla_a J^a = 0$ we obtain

$$\frac{\eta Q(D-2)}{f_0 r}\left[\frac{1}{\sqrt{g_0}}\left(\frac{1}{\sqrt{g_0}}\right)'\left(\frac{\dot{\Psi}_g'}{g_0'} - \frac{2\dot{\Psi}_g}{g_0} - \frac{\dot{\Psi}_f}{f_0} + \frac{\ddot{\Psi}_\Phi}{Q}\right)\right.$$
$$\left.+\frac{(D-3)(1-g_0)}{2rg_0}\left(\frac{\dot{\Psi}_g}{g_0(g_0-1)} - \frac{\dot{\Psi}_f}{f_0} + \frac{\ddot{\Psi}_\Phi}{Q}\right)\right] + \frac{\eta(D-2)F'}{2rf_0g_0^2}\ddot{\Psi}_g$$
$$+\left\{\frac{\eta F'(D-3)(D-2)}{2r^2 g_0}\left[\left(\frac{1}{g_0}-1\right)\left(\frac{\Psi_f'}{f_0'} - \frac{\Psi_g}{g_0} - \frac{\Psi_f}{f_0}\right) + \frac{\Psi_g}{g_0^2}\right]\right.$$
$$\left.+\frac{\eta(D-2)Q}{2f_0g_0^2}\dot{\Psi}_g + \frac{(D-2)}{r}\left[\frac{\sqrt{f_0}'}{\sqrt{f_0}g_0} + \frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right]\left(-\frac{\eta\Psi_\Phi'}{g_0} + \frac{F'\Psi_g}{g_0^2}\right)\right\}'$$
$$+\left(\frac{f_0'}{2f_0} + \frac{g_0'}{2g_0} + \frac{D-2}{r}\right)\left\{\frac{\eta F'(D-3)(D-2)}{2r^2 g_0}\left[\left(\frac{1}{g_0}-1\right)\left(\frac{\Psi_f'}{f_0'} - \frac{\Psi_g}{g_0} - \frac{\Psi_f}{f_0}\right) + \frac{\Psi_g}{g_0^2}\right]\right.$$
$$\left.+\frac{\eta(D-2)Q}{2f_0g_0^2}\dot{\Psi}_g + \frac{(D-2)}{r}\left[\frac{\sqrt{f_0}'}{\sqrt{f_0}g_0} + \frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right]\left(-\frac{\eta\Psi_\Phi'}{g_0} + \frac{F'\Psi_g}{g_0^2}\right)\right\}$$
$$+\left(\frac{\dot{\Psi}_f}{2f_0} + \frac{\dot{\Psi}_g}{2g_0}\right)\frac{\eta Q(D-2)}{f_0 r}\left(\frac{1}{\sqrt{g_0}}\left(\frac{1}{\sqrt{g_0}}\right)' + \frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right) = 0.$$

This equation should constraint the modified Einstein field equation.

2.2.4 H_{ab} components

Since the modified Einstein field equation in our case becomes $G_{ab} - H_{ab} = 0$, we need to show the nonzero components of H_{ab} as follows. The expressions contain both $O(1)$ and $O(\delta)$.

$\underline{H_{ab}^{(3)}:}$

$$H_{tt}^{(3)} = \left\{-\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0} - \frac{g_0'}{g_0}\right) + \left(1-\frac{1}{g_0}\right)\frac{(D-3)(D-2)}{2r^2}\right\}Q^2$$
$$+\delta\left[\frac{f_0''}{2f_0g_0}\left(-\frac{\Psi_f''}{f_0''} + \frac{\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right) + \frac{f_0'^2}{4f_0^2g_0}\left(\frac{2\Psi_f'}{f_0'} - \frac{2\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right)\right.$$
$$+\frac{f_0'g_0'}{4f_0g_0^2}\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi_g'}{g_0'} - \frac{\Psi_f}{f_0} - \frac{2\Psi_g}{g_0}\right)$$
$$-\frac{\ddot{\Psi}_g}{2f_0g_0} + \frac{(D-2)f_0'}{2rf_0g_0}\left(-\frac{\Psi_f'}{f_0'} + \frac{\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right) + \frac{(D-2)g_0'}{2rg_0^2}\left(\frac{\Psi_g'}{g_0'} - \frac{2\Psi_g}{g_0}\right) + \frac{(D-3)(D-2)}{2r^2}\frac{\Psi_g}{g_0^2}$$
$$\left.+\left\{-\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0} - \frac{g_0'}{g_0}\right) + \left(1-\frac{1}{g_0}\right)\frac{(D-3)(D-2)}{2r^2}\right\}\frac{2\Psi_g}{Q}\right]Q^2,$$

$$
\begin{aligned}
H_{tr}^{(3)} &= \left\{-\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right) + \left(1-\frac{1}{g_0}\right)\frac{(D-3)(D-2)}{2r^2}\right\}QF' \\
&+\delta\left[\frac{f_0''}{2f_0g_0}\left(-\frac{\Psi_f''}{f_0''}+\frac{\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right)+\frac{f_0'^2}{4f_0^2g_0}\left(\frac{2\Psi_f'}{f_0'}-\frac{2\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right)\right. \\
&+\frac{f_0'g_0'}{4f_0g_0^2}\left(\frac{\Psi_f'}{f_0'}+\frac{\Psi_g'}{g_0'}-\frac{\Psi_f}{f_0}-\frac{2\Psi_g}{g_0}\right) \\
&-\frac{\ddot{\Psi}_g}{2f_0g_0}+\frac{(D-2)f_0'}{2rf_0g_0}\left(-\frac{\Psi_f'}{f_0'}+\frac{\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right)+\frac{(D-2)g_0'}{2rg_0^2}\left(\frac{\Psi_g'}{g_0'}-\frac{2\Psi_g}{g_0}\right)+\frac{(D-3)(D-2)}{2r^2}\frac{\Psi_g}{g_0^2} \\
&\left.+\left\{-\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right) + \left(1-\frac{1}{g_0}\right)\frac{(D-3)(D-2)}{2r^2}\right\}\left(\frac{\dot{\Psi}_g}{Q}+\frac{\Psi_g'}{F'}\right)\right]QF',
\end{aligned}
$$

$$
\begin{aligned}
H_{rr}^{(3)} &= \left\{-\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right) + \left(1-\frac{1}{g_0}\right)\frac{(D-3)(D-2)}{2r^2}\right\}F'^2 \\
&+\delta\left[\frac{f_0''}{2f_0g_0}\left(-\frac{\Psi_f''}{f_0''}+\frac{\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right)+\frac{f_0'^2}{4f_0^2g_0}\left(\frac{2\Psi_f'}{f_0'}-\frac{2\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right)\right. \\
&+\frac{f_0'g_0'}{4f_0g_0^2}\left(\frac{\Psi_f'}{f_0'}+\frac{\Psi_g'}{g_0'}-\frac{\Psi_f}{f_0}-\frac{2\Psi_g}{g_0}\right) \\
&-\frac{\ddot{\Psi}_g}{2f_0g_0}+\frac{(D-2)f_0'}{2rf_0g_0}\left(-\frac{\Psi_f'}{f_0'}+\frac{\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right)+\frac{(D-2)g_0'}{2rg_0^2}\left(\frac{\Psi_g'}{g_0'}-\frac{2\Psi_g}{g_0}\right)+\frac{(D-3)(D-2)}{2r^2}\frac{\Psi_g}{g_0^2} \\
&\left.+\left\{-\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right) + \left(1-\frac{1}{g_0}\right)\frac{(D-3)(D-2)}{2r^2}\right\}\frac{2\Psi_g'}{F'}\right]F'^2.
\end{aligned}
$$

$\underline{H_{ab}^{(4)}}$:

$$
\begin{aligned}
H_{tt}^{(4)} &= -2Q^2\left\{\left(-\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}}-\frac{(D-2)f_0'}{2rf_0g_0}\right\} \\
&+\delta\left[\left\{\frac{f_0''}{2f_0g_0}\left(-\frac{\Psi_f''}{f_0''}+\frac{\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right)+\frac{f_0'^2}{4f_0^2g_0}\left(\frac{2\Psi_f'}{f_0'}-\frac{2\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right)\right.\right. \\
&+\frac{f_0'g_0'}{4f_0g_0^2}\left(\frac{\Psi_f'}{f_0'}+\frac{\Psi_g'}{g_0'}-\frac{\Psi_f}{f_0}-\frac{2\Psi_g}{g_0}\right) \\
&+\frac{\ddot{\Psi}_g}{2f_0g_0}+\frac{(D-2)f_0'}{2rf_0g_0}\left(-\frac{\Psi_f'}{f_0'}+\frac{\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right) \\
&\left.\left.+\left\{\left(-\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}}-\frac{(D-2)f_0'}{2rf_0g_0}\right\}\frac{2\dot{\Psi}_\Phi}{Q}\right\}(-2Q^2)-QF'\frac{(D-2)\dot{\Psi}_g}{rg_0^2}\right],
\end{aligned}
$$

$$H_{rr}^{(4)} = 2F'^2\left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)g_0'}{2rg_0^2}\right\}$$
$$+\delta\left[\left\{\frac{f_0''}{2f_0g_0}\left(\frac{\Psi_f''}{f_0''} - \frac{\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right) + \frac{f_0'^2}{4f_0^2g_0}\left(-\frac{2\Psi_f'}{f_0'} + \frac{2\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right)\right.\right.$$
$$+\frac{f_0'g_0'}{4f_0g_0^2}\left(-\frac{\Psi_f'}{f_0'} - \frac{\Psi_g'}{g_0'} + \frac{\Psi_f}{f_0} + \frac{2\Psi_g}{g_0}\right)$$
$$-\frac{\ddot{\Psi}_g}{2f_0g_0} + \frac{(D-2)g_0'}{2rg_0^2}\left(-\frac{\Psi_g'}{g_0'} + \frac{2\Psi_g}{g_0}\right)$$
$$\left.\left.+\left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}} - \frac{(D-2)g_0'}{2rg_0^2}\right\}\frac{2\Psi_\Phi'}{F'}\right\}2F'^2 - \frac{QF'}{f_0^2}\frac{(D-2)\dot{\Psi}_g}{rg_0}\right],$$

$$H_{tr}^{(4)} = \left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{2}{\sqrt{f_0g_0}} + \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0} - \frac{g_0'}{g_0}\right)\right\}QF'$$
$$+\delta\left[\left\{\frac{f_0''}{f_0g_0}\left(\frac{\Psi_f''}{f_0''} - \frac{\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right) + \frac{f_0'^2}{2f_0^2g_0}\left(-\frac{2\Psi_f'}{f_0'} + \frac{2\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right)\right.\right.$$
$$+\frac{f_0'g_0'}{2f_0g_0^2}\left(-\frac{\Psi_f'}{f_0'} - \frac{\Psi_g'}{g_0'} + \frac{\Psi_f}{f_0} + \frac{2\Psi_g}{g_0}\right)$$
$$-\frac{\ddot{\Psi}_g}{f_0g_0} + \frac{(D-2)f_0'}{2rf_0g_0}\left(\frac{\Psi_f'}{f_0'} - \frac{\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right) + \frac{(D-2)g_0'}{2rg_0^2}\left(-\frac{\Psi_g'}{g_0'} + \frac{2\Psi_g}{g_0}\right)$$
$$\left.\left.+\left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{2}{\sqrt{f_0g_0}} + \frac{(D-2)}{2rg_0}\left(\frac{f_0'}{f_0} - \frac{g_0'}{g_0}\right)\right\}\left(\frac{\dot{\Psi}_\Phi}{Q} + \frac{\Psi_\Phi'}{F'}\right)\right\}QF' + \left(\frac{Q^2}{f_0} - \frac{F'^2}{g_0}\right)\frac{(D-2)\dot{\Psi}_g}{2rg_0}\right].$$

<u>$H_{ab}^{(5)}$:</u>

$$H_{tt}^{(5)} = \left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}}\frac{F'^2f_0}{g_0}$$
$$+\delta\left[\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}}\left\{\frac{2\Psi_\Phi'}{F'} + \frac{\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right\} + \frac{f_0''}{2f_0g_0}\left(\frac{\Psi_f''}{f_0''} - \frac{\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right)\right.$$
$$\left.+\frac{f_0'^2}{4f_0^2g_0}\left(-\frac{2\Psi_f'}{f_0'} + \frac{2\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right) + \frac{f_0'g_0'}{4f_0g_0^2}\left(-\frac{\Psi_f'}{f_0'} - \frac{\Psi_g'}{g_0'} + \frac{\Psi_f}{f_0} + \frac{2\Psi_g}{g_0}\right) - \frac{\ddot{\Psi}_g}{2f_0g_0}\right]\frac{F'^2f_0}{g_0},$$

$$H_{tr}^{(5)} = \left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}}F'Q$$
$$+\delta\left[\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0g_0}}\left\{\frac{\dot{\Psi}_\Phi}{Q} + \frac{\Psi_\Phi'}{F'}\right\} + \frac{f_0''}{2f_0g_0}\left(\frac{\Psi_f''}{f_0''} - \frac{\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right)\right.$$
$$\left.+\frac{f_0'^2}{4f_0^2g_0}\left(-\frac{2\Psi_f'}{f_0'} + \frac{2\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right) + \frac{f_0'g_0'}{4f_0g_0^2}\left(-\frac{\Psi_f'}{f_0'} - \frac{\Psi_g'}{g_0'} + \frac{\Psi_f}{f_0} + \frac{2\Psi_g}{g_0}\right) - \frac{\ddot{\Psi}_g}{2f_0g_0}\right]F'Q,$$

$$
\begin{aligned}
H_{rr}^{(5)} &= \left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)' \frac{1}{\sqrt{f_0 g_0}} \frac{Q^2 g_0}{f_0} \\
&+ \delta\left[\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)' \frac{1}{\sqrt{f_0 g_0}}\left\{\frac{2\dot{\Psi}_\Phi}{Q} + \frac{\Psi_g}{g_0} - \frac{\Psi_f}{f_0}\right\} + \frac{f_0''}{2 f_0 g_0}\left(\frac{\Psi_f''}{f_0''} - \frac{\Psi_f}{f_0} - \frac{\Psi_g}{g_0}\right)\right. \\
&\left. + \frac{f_0'^2}{4 f_0^2 g_0}\left(-\frac{2\Psi_f'}{f_0'} + \frac{2\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right) + \frac{f_0' g_0'}{4 f_0 g_0^2}\left(-\frac{\Psi_f'}{f_0'} - \frac{\Psi_g'}{g_0'} + \frac{\Psi_f}{f_0} + \frac{2\Psi_g}{g_0}\right) - \frac{\ddot{\Psi}_g}{2 f_0 g_0}\right] \frac{Q^2 g_0}{f_0},
\end{aligned}
$$

$$
\begin{aligned}
H_{ij}^{(5)} &= g_{ij}\left\{-\frac{f_0'}{2 r f_0 g_0} \frac{Q^2}{f_0} - \frac{g_0'}{2 r g_0^2} \frac{F'^2}{g_0}\right\} \\
&+ \delta\left[g_{ij}\left\{\frac{f_0'}{2 r f_0 g_0} \frac{Q^2}{f_0}\left(-\frac{\Psi_f'}{f_0'} + \frac{2\Psi_f}{f_0} + \frac{\Psi_g}{g_0} - \frac{2\dot{\Psi}_\Phi}{Q}\right) + \frac{\dot{\Psi}_g}{r g_0} \frac{Q F'}{f_0 g_0}\right.\right. \\
&\left.\left. + \frac{g_0'}{2 r g_0^2} \frac{F'^2}{g_0}\left(-\frac{\Psi_g'}{g_0'} + \frac{3\Psi_g}{g_0} - \frac{2\Psi_\Phi'}{F'}\right)\right\}\right].
\end{aligned}
$$

<u>$H_{ab}^{(6)}$:</u>

$$
\begin{aligned}
H_{tt}^{(6)} &= \left\{\frac{1}{f_0}\left(\frac{f_0' F'}{2 g_0}\right)^2 - \frac{1}{g_0}\left(\frac{f_0' Q}{f_0}\right)^2\right\} \\
&+ \delta\left[\left\{-\frac{\Psi_f}{f_0^2}\left(\frac{f_0' F'}{2 g_0}\right)^2 + \frac{2}{f_0}\left(\frac{f_0' F'}{2 g_0}\right)^2\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{f_0' F'}{g_0} \frac{Q \dot{\Psi}_f}{2 f_0^2} - \frac{f_0' F'}{f_0 g_0} \ddot{\Psi}_\Phi\right.\right. \\
&\left.\left. + \frac{\Psi_g}{g_0^2}\left(\frac{f_0' Q}{f_0}\right)^2 + \frac{2}{g_0}\left(\frac{f_0' Q}{f_0}\right)^2\left(-\frac{\Psi_f'}{f_0'} - \frac{\dot{\Psi}_\Phi}{Q} + \frac{\Psi_f}{f_0}\right) - \frac{f_0' Q}{f_0 g_0} \frac{F' \dot{\Psi}_g}{g_0} + \frac{2 f_0' Q}{f_0 g_0} \dot{\Psi}_\Phi'\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{tr}^{(6)} &= \frac{f_0' Q}{f_0}\left\{\frac{f_0' F'}{2 f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2 g_0}\right)\right\} \\
&+ \delta\left[\left\{\frac{f_0' F'}{2 f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2 g_0}\right)\right\}\left[\frac{f_0' Q}{f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\dot{\Psi}_\Phi}{Q} - \frac{\Psi_f}{f_0}\right) + \frac{F' \dot{\Psi}_g}{2 g_0} - \dot{\Psi}_\Phi'\right]\right. \\
&+ \frac{f_0' Q}{f_0}\left\{-\frac{f_0' F'}{2 g_0} \frac{\Psi_f}{f_0^2} + \frac{f_0' F'}{2 g_0 f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{Q \dot{\Psi}_f}{2 f_0^2} - \frac{\ddot{\Psi}_\Phi}{f_0} + \frac{\Psi_g}{g_0^2}\left(\frac{g_0' F'}{2 g_0} - F''\right)\right. \\
&\left.\left. + \frac{g_0' F'}{2 g_0^2}\left(-\frac{\Psi_g'}{g_0'} - \frac{\Psi_\Phi'}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q \dot{\Psi}_g}{f_0 g_0} + \frac{\Psi_\Phi''}{g_0}\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{rr}^{(6)} &= \left\{\frac{1}{f_0}\left(\frac{f_0' Q}{f_0}\right)^2 - \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2 g_0}\right)^2\right\} \\
&+ \delta\left[\frac{2}{f_0}\left(\frac{f_0' Q}{f_0}\right)^2\left(\frac{\Psi_f'}{f_0'} + \frac{\dot{\Psi}_\Phi}{Q} - \frac{\Psi_f}{f_0}\right) + \frac{f_0' Q}{f_0^2} \frac{F' \dot{\Psi}_g}{g_0} - \frac{2 f_0' Q}{f_0^2} \dot{\Psi}_\Phi' - \left(\frac{f_0' Q}{f_0}\right)^2 \frac{\Psi_f}{f_0^2}\right. \\
&\left. + \left[\frac{g_0' F'}{g_0^2}\left(\frac{\Psi_g'}{g_0'} + \frac{\Psi_\Phi'}{F'} + \frac{\Psi_g}{g_0}\right) + \frac{2 Q \dot{\Psi}_g}{f_0 g_0} - \frac{2\Psi_\Phi''}{g_0}\right]\left(F'' - \frac{g_0' F'}{2 g_0}\right) + \frac{\Psi_g}{g_0^2}\left(F'' - \frac{g_0' F'}{2 g_0}\right)^2\right],
\end{aligned}
$$

$$H_{ij}^{(6)} = -g_{ij}\frac{F'^2}{r^2 g_0^2} + \delta\left[g_{ij}\frac{F'^2}{r^2 g_0^2}\left(\frac{2\Psi_g}{g_0} - \frac{2\Psi'_\Phi}{F'}\right)\right].$$

$\underline{H_{ab}^{(7)}}$:

$$\begin{aligned}
H_{ab}^{(7)} = &\left\{\frac{1}{f_0^2}\left(\frac{f_0'F'}{2g_0}\right)^2 + \frac{1}{g_0^2}\left(F'' - \frac{g_0'F'}{2g_0}\right)^2 + \frac{(D-2)}{r^2}\frac{F'^2}{g_0^2} - \frac{2}{f_0 g_0}\left(\frac{f_0'Q}{f_0}\right)^2\right\}\frac{1}{2}diag(-f_0, g_0, g_{ij})_{ab} \\
&+ \delta\left[\left\{\frac{1}{f_0^2}\left(\frac{f_0'F'}{2g_0}\right)^2 + \frac{1}{g_0^2}\left(F'' - \frac{g_0'F'}{2g_0}\right)^2 + \frac{(D-2)}{r^2}\frac{F'^2}{g_0^2} - \frac{2}{f_0 g_0}\left(\frac{f_0'Q}{f_0}\right)^2\right\}\frac{1}{2}diag(-\Psi_f, \Psi_g, 0)_{ab}\right. \\
&+ \left\{-\frac{2\Psi_f}{f_0^3}\left(\frac{f_0'F'}{2g_0}\right)^2 + \frac{2}{f_0^2}\left(\frac{f_0'F'}{2g_0}\right)^2\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi'_\Phi}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{f_0'F'}{g_0}\frac{Q\dot{\Psi}_f}{2f_0^3} - \frac{f_0'F'}{f_0^2 g_0}\ddot{\Psi}_\Phi\right. \\
&+ \left[\frac{g_0'F'}{g_0^3}\left(\frac{\Psi_g'}{g_0'} + \frac{\Psi'_\Phi}{F'} + \frac{\Psi_g}{g_0}\right) + \frac{2Q\dot{\Psi}_g}{f_0 g_0^2} - \frac{2\Psi''_\Phi}{g_0^2}\right]\left(\frac{g_0'F'}{2g_0} - F''\right) - \frac{2\Psi_g}{g_0^3}\left(F'' - \frac{g_0'F'}{2g_0}\right)^2 \\
&+ \frac{(D-2)F'^2}{r^2 g_0^2}\left(\frac{2\Psi'_\Phi}{F'} - \frac{2\Psi_g}{g_0}\right) + \frac{2}{f_0 g_0}\left\{\frac{\Psi_f}{f_0} + \frac{\Psi_g}{g_0}\right\}\left(\frac{f_0'Q}{f_0}\right)^2 \\
&\left.\left.+ \frac{2}{f_0 g_0}\left[2\left(\frac{f_0'Q}{f_0}\right)^2\left(-\frac{\Psi_f'}{f_0'} - \frac{\dot{\Psi}_\Phi}{Q} + \frac{\Psi_f}{f_0}\right) - \frac{f_0'Q}{f_0}\frac{F'\dot{\Psi}_g}{g_0} + \frac{2f_0'Q}{f_0}\dot{\Psi}'_\Phi\right]\right\}\frac{1}{2}diag(-f_0, g_0, g_{ij})_{ab}\right].
\end{aligned}$$

$\underline{H_{ab}^{(8)}}$:

$$\begin{aligned}
H_{ab}^{(8)} = &\left\{\frac{f_0'F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}^2\frac{1}{2}diag(f_0, -g_0, -g_{ij})_{ab} \\
&+ \delta\left[\left\{\frac{f_0'F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}^2\frac{1}{2}diag(\Psi_f, -\Psi_g, 0)_{ab}\right. \\
&+ \left[-\frac{f_0'F'}{2g_0}\frac{\Psi_f}{f_0^2} + \frac{f_0'F'}{2g_0 f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi'_\Phi}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{Q\dot{\Psi}_f}{2f_0^2} - \frac{\ddot{\Psi}_\Phi}{f_0} + \frac{\Psi_g}{g_0^2}\left(\frac{g_0'F'}{2g_0} - F''\right)\right. \\
&\left.+ \frac{g_0'F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'} - \frac{\Psi'_\Phi}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_g}{f_0 g_0} + \frac{\Psi''_\Phi}{g_0} + \frac{(D-2)F'}{rg_0}\left(\frac{\Psi'_\Phi}{F'} - \frac{\Psi_g}{g_0}\right)\right] \\
&\left.\times\left\{\frac{f_0'F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}diag(f_0, -g_0, -g_{ij})_{ab}\right].
\end{aligned}$$

$\underline{H_{ab}^{(9)}}$:

$$\begin{aligned}
H_{tt}^{(9)} = &-\frac{F'f_0'}{2g_0}\left\{\frac{f_0'F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\} \\
&+ \delta\left[\left\{\frac{f_0'F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}\left[\frac{f_0'F'}{2g_0}\left(-\frac{\Psi_f'}{f_0'} - \frac{\Psi'_\Phi}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_f}{2f_0} + \ddot{\Psi}_\Phi\right]\right. \\
&- \frac{F'f_0'}{2g_0}\left\{-\frac{f_0'F'}{2g_0}\frac{\Psi_f}{f_0^2} + \frac{f_0'F'}{2g_0 f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi'_\Phi}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{Q\dot{\Psi}_f}{2f_0^2} - \frac{\ddot{\Psi}_\Phi}{f_0} + \frac{\Psi_g}{g_0^2}\left(\frac{g_0'F'}{2g_0} - F''\right)\right. \\
&\left.\left.+ \frac{g_0'F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'} - \frac{\Psi'_\Phi}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_g}{f_0 g_0} + \frac{\Psi''_\Phi}{g_0} + \frac{(D-2)F'}{rg_0}\left(\frac{\Psi'_\Phi}{F'} - \frac{\Psi_g}{g_0}\right)\right\}\right],
\end{aligned}$$

$$
\begin{aligned}
H_{tr}^{(9)} = & \frac{f_0' Q}{f_0}\left\{\frac{f_0' F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\} \\
& + \delta\left[\left\{\frac{f_0' F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}\left[-\frac{f_0' Q}{f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\dot{\Psi}_\Phi}{Q} - \frac{\Psi_f}{f_0}\right) - \frac{F'\dot{\Psi}_g}{2g_0} + \dot{\Psi}_\Phi'\right]\right. \\
& - \frac{f_0' Q}{f_0}\left\{-\frac{f_0' F'}{2g_0}\frac{\Psi_f}{f_0^2} + \frac{f_0' F'}{2g_0 f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{Q\dot{\Psi}_f}{2f_0^2} - \frac{\ddot{\Psi}_\Phi}{f_0} + \frac{\Psi_g}{g_0^2}\left(\frac{g_0' F'}{2g_0} - F''\right)\right. \\
& \left.\left. + \frac{g_0' F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'} - \frac{\Psi_\Phi'}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_g}{f_0 g_0} + \frac{\Psi_\Phi''}{g_0} + \frac{(D-2)F'}{rg_0}\left(\frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right)\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{rr}^{(9)} = & \left\{F'' - \frac{g_0' F'}{2g_0}\right\}\left\{\frac{f_0' F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\} \\
& + \delta\left[\left\{\frac{f_0' F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}\left[\frac{g_0' F'}{2g_0}\left(-\frac{\Psi_g'}{g_0'} - \frac{\Psi_\Phi'}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_g}{f_0} + \Psi_\Phi''\right]\right. \\
& + \left\{F'' - \frac{g_0' F'}{2g_0}\right\}\left\{-\frac{f_0' F'}{2g_0}\frac{\Psi_f}{f_0^2} + \frac{f_0' F'}{2g_0 f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{Q\dot{\Psi}_f}{2f_0^2} - \frac{\ddot{\Psi}_\Phi}{f_0} + \frac{\Psi_g}{g_0^2}\left(\frac{g_0' F'}{2g_0} - F''\right)\right. \\
& \left.\left. + \frac{g_0' F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'} - \frac{\Psi_\Phi'}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_g}{f_0 g_0} + \frac{\Psi_\Phi''}{g_0} + \frac{(D-2)F'}{rg_0}\left(\frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right)\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{ij}^{(9)} = & g_{ij}\frac{F'}{rg_0}\left\{\frac{f_0' F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\} \\
& + \delta\left[\left\{\frac{f_0' F'}{2f_0 g_0} + \frac{1}{g_0}\left(F'' - \frac{g_0' F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\} g_{ij}\frac{F'}{rg_0}\left(\frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right)\right. \\
& + g_{ij}\frac{F'}{rg_0}\left\{-\frac{f_0' F'}{2g_0}\frac{\Psi_f}{f_0^2} + \frac{f_0' F'}{2g_0 f_0}\left(\frac{\Psi_f'}{f_0'} + \frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right) + \frac{Q\dot{\Psi}_f}{2f_0^2} - \frac{\ddot{\Psi}_\Phi}{f_0} + \frac{\Psi_g}{g_0^2}\left(\frac{g_0' F'}{2g_0} - F''\right)\right. \\
& \left.\left. + \frac{g_0' F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'} - \frac{\Psi_\Phi'}{F'} + \frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_g}{f_0 g_0} + \frac{\Psi_\Phi''}{g_0} + \frac{(D-2)F'}{rg_0}\left(\frac{\Psi_\Phi'}{F'} - \frac{\Psi_g}{g_0}\right)\right\}\right].
\end{aligned}
$$

$\underline{H_{ab}^{(10)}:}$

$$
\begin{aligned}
H_{tt}^{(10)} =& \frac{(D-2)f_0}{r}\left\{\frac{g_0'}{2g_0^2}-\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\frac{1}{2}\left\{\frac{f_0'F'}{2f_0g_0}+\frac{1}{g_0}\left(F''-\frac{g_0'F'}{2g_0}\right)+\frac{(D-2)F'}{rg_0}\right\}\\
&+\delta\left[\frac{(D-2)f_0}{r}\left\{\frac{g_0'}{2g_0^2}-\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\right.\\
&\times\frac{1}{2}\left\{-\frac{f_0'F'}{2g_0}\frac{\Psi_f}{f_0^2}+\frac{f_0'F'}{2g_0f_0}\left(\frac{\Psi_f'}{f_0'}+\frac{\Psi_\Phi'}{F'}-\frac{\Psi_g}{g_0}\right)+\frac{Q\dot{\Psi}_f}{2f_0^2}-\frac{\ddot{\Psi}_\Phi}{f_0}+\frac{\Psi_g}{g_0^2}\left(\frac{g_0'F'}{2g_0}-F''\right)\right.\\
&\left.+\frac{g_0'F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'}-\frac{\Psi_\Phi'}{F'}+\frac{\Psi_g}{g_0}\right)-\frac{Q\dot{\Psi}_g}{f_0g_0}+\frac{\Psi_\Phi''}{g_0}+\frac{(D-2)F'}{rg_0}\left(\frac{\Psi_\Phi'}{F'}-\frac{\Psi_g}{g_0}\right)\right\}\\
&+\frac{(D-2)f_0}{r}\left\{\frac{\Psi_g'}{2g_0^2}-\frac{g_0'\Psi_g}{g_0^3}+\frac{(D-3)}{2r}\frac{\Psi_g}{g_0^2}+\frac{\Psi_f}{f_0}\left\{\frac{g_0'}{2g_0^2}-\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\right\}\\
&\left.\times\frac{1}{2}\left\{\frac{f_0'F'}{2f_0g_0}+\frac{1}{g_0}\left(F''-\frac{g_0'F'}{2g_0}\right)+\frac{(D-2)F'}{rg_0}\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{rr}^{(10)} =& \frac{(D-2)g_0}{r}\left\{\frac{f_0'}{2f_0g_0}+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\frac{1}{2}\left\{\frac{f_0'F'}{2f_0g_0}+\frac{1}{g_0}\left(F''-\frac{g_0'F'}{2g_0}\right)+\frac{(D-2)F'}{rg_0}\right\}\\
&+\delta\left[\frac{(D-2)g_0}{r}\left\{\frac{f_0'}{2f_0g_0}+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\right.\\
&\times\frac{1}{2}\left\{-\frac{f_0'F'}{2g_0}\frac{\Psi_f}{f_0^2}+\frac{f_0'F'}{2g_0f_0}\left(\frac{\Psi_f'}{f_0'}+\frac{\Psi_\Phi'}{F'}-\frac{\Psi_g}{g_0}\right)+\frac{Q\dot{\Psi}_f}{2f_0^2}-\frac{\ddot{\Psi}_\Phi}{f_0}+\frac{\Psi_g}{g_0^2}\left(\frac{g_0'F'}{2g_0}-F''\right)\right.\\
&\left.+\frac{g_0'F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'}-\frac{\Psi_\Phi'}{F'}+\frac{\Psi_g}{g_0}\right)-\frac{Q\dot{\Psi}_g}{f_0g_0}+\frac{\Psi_\Phi''}{g_0}+\frac{(D-2)F'}{rg_0}\left(\frac{\Psi_\Phi'}{F'}-\frac{\Psi_g}{g_0}\right)\right\}\\
&+\frac{(D-2)g_0}{r}\left\{\frac{f_0'}{2f_0g_0}\left(\frac{\Psi_f'}{f_0'}-\frac{\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right)-\frac{(D-3)}{2r}\frac{\Psi_g}{g_0^2}+\frac{\Psi_g}{g_0}\left\{\frac{f_0'}{2f_0g_0}+\frac{(D-3)}{2r}\left(\frac{1}{g_0}-1\right)\right\}\right\}\\
&\left.\times\frac{1}{2}\left\{\frac{f_0'F'}{2f_0g_0}+\frac{1}{g_0}\left(F''-\frac{g_0'F'}{2g_0}\right)+\frac{(D-2)F'}{rg_0}\right\}\right],
\end{aligned}
$$

$$
\begin{aligned}
H_{ij}^{(10)} &= g_{ij}\left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0 g_0}} + \frac{(D-3)}{2rg_0}\left(\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right) + \left(\frac{1}{g_0}-1\right)\frac{(D-3)(D-4)}{2r^2}\right\}\frac{1}{2}\\
&\times\left\{\frac{f_0'F'}{2f_0g_0} + \frac{1}{g_0}\left(F''-\frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}\\
&+\delta\left[g_{ij}\left\{-\frac{\ddot{\Psi}_g}{2f_0g_0} + \frac{f_0''}{2f_0g_0}\left(\frac{\Psi_f''}{f_0''}-\frac{\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right) - \frac{f_0'^2}{4f_0^2g_0}\left(-\frac{2\Psi_f'}{f_0'}+\frac{2\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right)\right.\right.\\
&\left.+\frac{f_o'g_0'}{4f_0g_0^2}\left(-\frac{\Psi_f'}{f_0'}-\frac{\Psi_g'}{g_0'}+\frac{\Psi_f}{f_0}+\frac{2\Psi_g}{g_0}\right) + \frac{(D-3)g_0'}{2rg_0^2}\left(-\frac{\Psi_g'}{g_0'}+\frac{2\Psi_g}{g_0}\right) + \frac{(D-3)(D-4)}{2r^2}\frac{\Psi_g}{g_0^2}\right\}\\
&\times\frac{1}{2}\left\{\frac{f_0'F'}{2f_0g_0} + \frac{1}{g_0}\left(F''-\frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}\\
&+g_{ij}\left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{\sqrt{f_0 g_0}} + \frac{(D-3)}{2rg_0}\left(\frac{f_0'}{f_0}-\frac{g_0'}{g_0}\right) + \left(\frac{1}{g_0}-1\right)\frac{(D-3)(D-4)}{2r^2}\right\}\\
&\times\frac{1}{2}\left\{-\frac{f_0'F'}{2g_0}\frac{\Psi_f}{f_0^2} + \frac{f_0'F'}{2g_0f_0}\left(\frac{\Psi_f'}{f_0'}+\frac{\Psi_\Phi'}{F'}-\frac{\Psi_g}{g_0}\right) + \frac{Q\dot{\Psi}_f}{2f_0^2} - \frac{\ddot{\Psi}_\Phi}{f_0} + \frac{\Psi_g}{g_0^2}\left(\frac{g_0'F'}{2g_0}-F''\right)\right.\\
&\left.\left.+\frac{g_0'F'}{2g_0^2}\left(-\frac{\Psi_g'}{g_0'}-\frac{\Psi_\Phi'}{F'}+\frac{\Psi_g}{g_0}\right) - \frac{Q\dot{\Psi}_g}{f_0g_0} + \frac{\Psi_\Phi''}{g_0} + \frac{(D-2)F'}{rg_0}\left(\frac{\Psi_\Phi'}{F'}-\frac{\Psi_g}{g_0}\right)\right\}\right],
\end{aligned}
$$

$$
H_{tr}^{(10)} = \delta\frac{(D-2)\dot{\Psi}_g}{4rg_0}\left\{\frac{f_0'F'}{2f_0g_0} + \frac{1}{g_0}\left(F''-\frac{g_0'F'}{2g_0}\right) + \frac{(D-2)F'}{rg_0}\right\}.
$$

$\underline{H_{ab}^{(11)}}$:

$$
\begin{aligned}
H_{ab}^{(11)} &= \left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{f_0g_0}\left[\frac{Q^2}{f_0}-\frac{F'^2}{g_0}\right] + \frac{(D-2)f_0'}{2rf_0g_0}\frac{Q^2}{f_0} + \frac{(D-2)g_0'}{2rg_0^2}\frac{F'^2}{g_0}\right\} diag(-f_0, g_0, g_{ij})_{ab}\\
&+\delta\left[\left\{\left(\frac{\sqrt{f_0}'}{\sqrt{g_0}}\right)'\frac{1}{f_0g_0}\left[\frac{Q^2}{f_0}-\frac{F'^2}{g_0}\right] + \frac{(D-2)f_0'}{2rf_0g_0}\frac{Q^2}{f_0} + \frac{(D-2)g_0'}{2rg_0^2}\frac{F'^2}{g_0}\right\} diag(-\Psi_f, \Psi_g, 0)_{ab}\right.\\
&+\left\{\left[\frac{Q^2}{f_0}-\frac{F'^2}{g_0}\right]\left[-\frac{\ddot{\Psi}_g}{2f_0g_0} + \frac{f_0''}{2f_0g_0}\left(\frac{\Psi_f''}{f_0''}-\frac{\Psi_f}{f_0}-\frac{\Psi_g}{g_0}\right)\right.\right.\\
&\left.+\frac{f_o'^2}{4f_0^2g_0}\left(-\frac{2\Psi_f'}{f_0'}+\frac{2\Psi_f}{f_0}+\frac{\Psi_g}{g_0}\right) + \frac{f_0'g_0'}{4f_0g_0^2}\left(-\frac{\Psi_f'}{f_0'}-\frac{\Psi_g'}{g_0'}+\frac{\Psi_f}{f_0}+\frac{2\Psi_g}{g_0}\right)\right]\\
&+\frac{(D-2)f_0'}{2rf_0g_0}\frac{Q^2}{f_0}\left(\frac{2\dot{\Psi}_\Phi}{Q}-\frac{2\Psi_f}{f_0}+\frac{\Psi_f'}{f_0'}-\frac{\Psi_g}{g_0}\right)\\
&\left.\left.+\frac{(D-2)g_0'}{2rg_0^2}\frac{F'^2}{g_0}\left(\frac{2\Psi_\Phi'}{F'}+\frac{\Psi_g'}{g_0'}-\frac{3\Psi_g}{g_0}\right)\right\} diag(-f_0, g_0, g_{ij})_{ab}\right].
\end{aligned}
$$

3 Discussion

Here our goal is to see the pattern for $O(\delta)$ of $\nabla_a J^a = 0$ and $G_{ab} = H_{ab}$. For the scalar current, we obtain this pattern

$$
\nabla_a J^a = \nabla_a J^a(\ddot{\Psi}_g, \Psi_f'', \ddot{\Psi}_\Phi, \Psi_\Phi'') = 0.
$$

For the modified Einstein field equation, see Table 1 below for each tensor components. Observe that, from $G_{ab} = H_{ab}$, we can have wave equation for scalar field perturbation:

$$A\,(B\Psi'_\Phi)' - C\ddot{\Psi}_\Phi + D\Psi_\Phi = 0.$$

However, there perturbation for the metric functions fails to follow similar pattern, because we may have the following pattern only:

$$A\left(B\Psi'_f\right)' - C\ddot{\Psi}_g + D\Psi_\Phi + E\Psi_f + I\Psi_g = 0.$$

This seems that the metric functions Ψ_f and Ψ_g cannot be decoupled because there exists neither $\ddot{\Psi}_f$ nor Ψ''_g. Further investigations are needed to encounter this problem. Below, we state a possible solution from the literature.

Table 1. Pattern for the highest derivative in the expression for each tensor components.

Components:	tt	rr	ij	tr
G_{ab}	Ψ'_g	Ψ'_f	$\ddot{\Psi}_g, \Psi''_f$	$\dot{\Psi}_g$
$H^{(3)}_{ab}$	$\ddot{\Psi}_g, \Psi''_f$	$\ddot{\Psi}_g, \Psi''_f$	-	$\ddot{\Psi}_g, \Psi''_f$
$H^{(4)}_{ab}$	$\ddot{\Psi}_g, \Psi''_f$	$\ddot{\Psi}_g, \Psi''_f$	-	$\ddot{\Psi}_g, \Psi''_f$
$H^{(5)}_{ab}$	$\ddot{\Psi}_g, \Psi''_f$	$\ddot{\Psi}_g, \Psi''_f$	$\Psi'_f, \dot{\Psi}_g, \Psi'_g, \dot{\Psi}_\Phi, \Psi'_\Phi$	$\ddot{\Psi}_g, \Psi''_f$
$H^{(6)}_{ab}$	$\ddot{\Psi}_\Phi$	Ψ''_Φ	-	$\ddot{\Psi}_\Phi, \Psi''_\Phi$
$H^{(7)}_{ab}$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	-
$H^{(8)}_{ab}$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	-
$H^{(9)}_{ab}$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$
$H^{(10)}_{ab}$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_\Phi, \Psi''_\Phi$	$\ddot{\Psi}_g, \Psi''_f, \ddot{\Psi}_\Phi, \Psi''_\Phi$	$\dot{\Psi}_g$
$H^{(11)}_{ab}$	$\ddot{\Psi}_g, \Psi''_f$	$\ddot{\Psi}_g, \Psi''_f$	$\ddot{\Psi}_g, \Psi''_f$	-
$G_{ab} - H_{ab}$	$\Psi''_f, \ddot{\Psi}_g, \ddot{\Psi}_\Phi, \Psi''_\Phi$	$\Psi''_f, \ddot{\Psi}_g, \ddot{\Psi}_\Phi, \Psi''_\Phi$	$\Psi''_f, \ddot{\Psi}_g, \ddot{\Psi}_\Phi, \Psi''_\Phi$	$\Psi''_f, \ddot{\Psi}_g, \ddot{\Psi}_\Phi, \Psi''_\Phi$

A possible solution from the literature is stated as follows. It seems that we need another form of expansion, such as the ones used by Chandrasekhar to derive the Zerilli equation. He used

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\varphi - q_2 dr - q_3 d\theta - \omega dt)^2 + e^{2\mu_2}dr^2 + e^{2\mu_3}d\theta^2,$$

with all functions are dependent on t, r, and θ. Then the metric functions are expanded as,

$$\begin{aligned}
\nu &\to \nu + e^{i\sigma t}N(r)P_l,\\
\mu_2 &\to \mu_2 + e^{i\sigma t}L(r)P_l,\\
\mu_3 &\to \mu_3 + e^{i\sigma t}[T(r)P_l + V(r)P_{l,\theta,\theta}],\\
\psi &\to \psi + e^{i\sigma t}[T(r)P_l + V(r)P_{l,\theta}\cot\theta],
\end{aligned}$$

with $P_l = P_l(\cos\theta)$ and $f_{,\theta} = \partial f/\partial\theta$. Then, using algebraic combinations of N, L, T, and V, the Zerilli equation can be obtained. We observe that this means we need to expand our metric function $h(r)$ as $h(r) = 1 + \delta\Psi_h(t, r)$. Observe that we may follow Chandrasekhar's derivation since $P_0 = 1$, which is aligned to our case where all functions are independent from θ.

4 Conclusion

In this work, we attempt to perturb the black-hole solution in a certain higher-dimensional space-time. Our goal here is to obtain the equations of motion for the perturbation terms similar to the Regge-Wheeler-Zerilli equation. We have not yet succeeded in finding them. It seems that we need to choose a different expansion or some kind of algebraic relations to simplify these equations that came from the modified Einstein field equation. One possibility is the derivation of the Zerilli equation from Chandrasekhar's book. We shall report this in a future paper.

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