Advancing Curve and Surface Images Modeling with Two-Parameters Polynomial-Based Quaternary Subdivision Schemes

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Abstract. In the context of this paper, we introduce a novel polynomial function that relies on two parameters. This polynomial enables the creation of a family of quaternary subdivision schemes for curve and surface modeling. One of these parameters is responsible for determining the specific member of the family while the other parameter provides the means to finely control the shape of the resulting curve or the regular surface images. This two-parameter approach adds significant versatility to the subdivision schemes to meet specific requirements and preferences. The exploration of various family members within this class of quaternary schemes is a focal point of our research. By adjusting the parameters, we investigate and delineate the distinctive characteristics of specific family members. This provides valuable insights into how these schemes can be harnessed to achieve various modeling goals. This insight empowers users to select the most suitable family members in accordance with their specific needs and design objectives.

1 Introduction

Quaternary subdivision schemes are sophisticated techniques used to generate highly precise and refined approximations for curves and surfaces. These schemes initiate with a coarse representation known as the control polygon and progressively enhance this polygon. Ultimately resulting in a smooth and refined shape as the process approaches its limit. The quaternary scheme achieves faster topological refinement compared to the binary and ternary ones. Therefore, opting for the quaternary scheme results in a quicker generation of curves and surfaces. The fundamental principle underlying quaternary subdivision schemes revolves

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around four distinct rules. Each rule introduces a new control point thereby refining the course polygon. Mathematically, given a sequence of control points $\{f_i^k \in \mathbb{R}^{\mathbb{N}}, i \in \mathbb{Z}, N \geq 2\}$, where $k \geq 0$ indicates the subdivision level, a quaternary subdivision scheme [1, 2] that generate a sequence of new points is defined as:

$$f_{4i+\alpha}^{k+1} = \sum_{j\in\mathbb{Z}} a_{\alpha,j} f_{i+j}^{k} \quad \alpha = 0, 1, 2, 3,$$

and $\sum_{j\in\mathbb{Z}} a_{\alpha,j} = 1$. Visually, this process is depicted in Figure 1. In this figure, the solid circles can be considered as points f_i^k, while the asterisks (cross marks) represent the new points f_i^{k+1}.
In 2013, Ghaffar et al. [3] proposed a generalized formula for generating 4-point subdivision

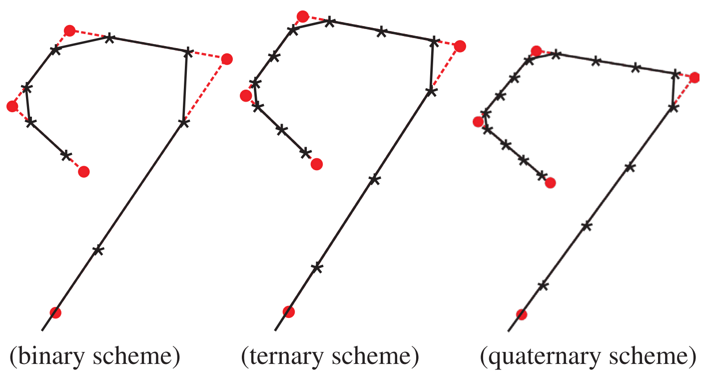


Figure 1. This figure illustrates how the binary, ternary and quaternary schemes operates. Here the solid circles show the course points while * shows refined/new points.

schemes encompassing binary, ternary, and quaternary subdivision schemes. In the same year, Amat and Liandrat [4] introduced a 4-point scheme designed to eliminate the Gibbs phenomenon. Pervez [5] presented a 4-point quaternary scheme while Mustafa et al. [6] introduced a family of multiparameter quaternary schemes. Siddiqi and Younis [8] introduced a family of quaternary schemes. They utilized the Cox-De Boor recursion formula for constructing these quaternary schemes. Nawaz et al. [9] and Tariq et al. [10] presented 7-point quaternary and 3-point quaternary schemes respectively.
This paper introduces a family of m-point quaternary schemes based on polynomials with two parameters. The distinct roles of these parameters are thoroughly discussed. A range of practical applications where these quaternary subdivision schemes prove to be invaluable are discussed. We illustrate how these schemes can be effectively applied to both curve and surface modeling. The paper serves as a comprehensive guide to the theory and practical implementation of these quaternary subdivision schemes.

2 Two-parameters polynomial-based family of quaternary schemes

In this section, a polynomial denoted as $\gamma^{m,w}(z)$ is introduced with two parameters namely, m and w. A family of quaternary schemes is derived from this polynomial. The parameter m is responsible for determining the specific family members from the polynomial while the parameter w plays a significant role in controlling the shapes of the resulting curve or surface when these schemes are applied. The novel polynomial is presented below.

$$\begin{aligned}\gamma^{m,w}(z) &= \frac{2^{m+2}\Big(z^3+z^2+z+1\Big)^{m+1}\Big(1+z\Big)^{m-1}}{(4z^3)^{2m+1}}\Big(\frac{1}{6}+w+\Big(\frac{1}{3}-2w\Big)z \\ &\quad +\Big(\frac{1}{2}+w\Big)z^2+\Big(\frac{1}{2}+w\Big)z^3+\Big(\frac{1}{3}-2w\Big)z^4+\Big(\frac{1}{6}+w\Big)z^5\Big).\end{aligned} \tag{1}$$

The polynomial above serves as the foundation for a set of quaternary subdivision schemes. As an illustration, two members of this family are provided below.

2.1 The 4-point and 5-point quaternary schemes

In this section, two schemes are derived as examples using the polynomial (1). Other members of the family of schemes can be obtained by following a similar procedure. Assuming $m = 2$, the following polynomial can be obtained:

$$\begin{aligned}\gamma^{2,w}(z) &= \Big(\frac{z^3+z^2+z+1}{4}\Big)^3\Big[\Big(\frac{1}{6}+w\Big)z^6+\Big(\frac{1}{2}-w\Big)z^5+\Big(\frac{5}{6}-w\Big)z^4 \\ &\quad +\Big(1+2w\Big)z^3+\Big(\frac{5}{6}-w\Big)z^2+\Big(\frac{1}{2}-w\Big)z+\Big(\frac{1}{6}+w\Big)\Big].\end{aligned} \tag{2}$$

This implies

$$\begin{aligned}\gamma^{2,w}(z) &= \Big(\frac{1}{384}+\frac{w}{64}\Big)+\Big(\frac{1}{64}+\frac{w}{32}\Big)z+\Big(\frac{5}{96}+\frac{w}{32}\Big)z^2+\Big(\frac{49}{384}+\frac{3w}{64}\Big)z^3 \\ &\quad +\Big(\frac{95}{384}+\frac{w}{64}\Big)z^4+\Big(\frac{19}{48}-\frac{w}{32}\Big)z^5+\Big(\frac{103}{192}-\frac{w}{32}\Big)z^6 \\ &\quad +\Big(\frac{239}{384}-\frac{5w}{64}\Big)z^7+\Big(\frac{239}{384}-\frac{5w}{64}\Big)z^8+\Big(\frac{103}{192}-\frac{w}{32}\Big)z^9 \\ &\quad +\Big(\frac{19}{48}-\frac{w}{32}\Big)z^{10}+\Big(\frac{95}{384}+\frac{w}{64}\Big)z^{11}+\Big(\frac{49}{384}+\frac{3w}{64}\Big)z^{12} \\ &\quad +\Big(\frac{5}{96}+\frac{w}{32}\Big)z^{13}+\Big(\frac{1}{64}+\frac{w}{32}\Big)z^{14}+\Big(\frac{1}{384}+\frac{w}{64}\Big)z^{15}.\end{aligned} \tag{3}$$

The coefficients of $z^0, z^1, \ldots, z^{15}$ in the polynomial (3) yield the following four rules:

$$\begin{cases} f^{k+1}_{4i} = \Big(\frac{1}{384}+\frac{w}{64}\Big)f^k_{i-2}+\Big(\frac{95}{384}+\frac{w}{64}\Big)f^k_{i-1}+\Big(\frac{239}{384}-\frac{5w}{64}\Big)f^k_i+\Big(\frac{49}{384}+\frac{3w}{64}\Big)f^k_{i+1} \\ f^{k+1}_{4i+1} = \Big(\frac{1}{64}+\frac{w}{32}\Big)f^k_{i-2}+\Big(\frac{19}{48}-\frac{w}{32}\Big)f^k_{i-1}+\Big(\frac{103}{192}-\frac{w}{32}\Big)f^k_i+\Big(\frac{5}{96}+\frac{w}{32}\Big)f^k_{i+1} \\ f^{k+1}_{4i+2} = \Big(\frac{5}{96}+\frac{w}{32}\Big)f^k_{i-2}+\Big(\frac{103}{192}-\frac{w}{32}\Big)f^k_{i-1}+\Big(\frac{19}{48}-\frac{w}{32}\Big)f^k_i+\Big(\frac{1}{64}+\frac{w}{32}\Big)f^k_{i+1} \\ f^{k+1}_{4i+3} = \Big(\frac{49}{384}+\frac{3w}{64}\Big)f^k_{i-2}+\Big(\frac{239}{384}-\frac{5w}{64}\Big)f^k_{i-1}+\Big(\frac{95}{384}+\frac{w}{64}\Big)f^k_i+\Big(\frac{1}{384}+\frac{w}{64}\Big)f^k_{i+1}. \end{cases} \tag{4}$$

The set of rules described above is commonly referred to as the 4-point quaternary subdivision scheme. In these rules, f^k_i represents the control points at the kth level while f^k_{i+1} represents the refined points at the $(k+1)$th level obtained from the 4-points at the kth level.

Similarly, by assuming $m = 3$, the 5-point quaternary subdivision scheme can be obtained:

$$\begin{cases} f_{4i}^{k+1} = \left(\frac{1}{3072} + \frac{w}{512}\right)f_{i-2}^k + \left(\frac{41}{512} + \frac{w}{32}\right)f_{i-1}^k + \left(\frac{191}{384} - \frac{11w}{256}\right)f_i^k + \left(\frac{199}{512} - \frac{w}{64}\right)f_{i+1}^k \\ \qquad + \left(\frac{103}{3072} + \frac{13w}{512}\right)f_{i+2}^k \\ f_{4i+1}^{k+1} = \left(\frac{1}{384} + \frac{w}{128}\right)f_{i-2}^k + \left(\frac{81}{512} + \frac{3w}{128}\right)f_{i-1}^k + \left(\frac{863}{1536} - \frac{7w}{128}\right)f_i^k + \left(\frac{409}{1536} + \frac{w}{128}\right)f_{i+1}^k \\ \qquad + \left(\frac{17}{1536} + \frac{w}{64}\right)f_{i+2}^k \\ f_{4i+2}^{k+1} = \left(\frac{17}{1536} + \frac{w}{64}\right)f_{i-2}^k + \left(\frac{409}{1536} + \frac{w}{128}\right)f_{i-1}^k + \left(\frac{863}{1536} - \frac{7w}{128}\right)f_i^k + \left(\frac{81}{512} + \frac{3w}{128}\right)f_{i+1}^k \\ \qquad + \left(\frac{1}{384} + \frac{w}{128}\right)f_{i+2}^k \\ f_{4i+3}^{k+1} = \left(\frac{103}{3072} + \frac{13w}{512}\right)f_{i-2}^k + \left(\frac{199}{512} - \frac{w}{64}\right)f_{i-1}^k + \left(\frac{191}{384} - \frac{11w}{256}\right)f_i^k + \left(\frac{41}{512} + \frac{w}{32}\right)f_{i+1}^k \\ \qquad + \left(\frac{1}{3072} + \frac{w}{512}\right)f_{i+2}^k. \end{cases} \tag{5}$$

3 Analysis of the 4-point and 5-point quaternary schemes

A concise analysis of the 4-point and 5-point quaternary schemes is presented here. A similar procedure can be applied to analyze other members of the family. Both schemes meet the necessary convergence conditions, as the sum of coefficients in all individual rules for both schemes is equal to one. The analysis procedure of [7] is employed to discuss the schemes.

Theorem 3.1 *The 4-point quaternary subdivision scheme exhibits C^0-continuity within the interval $-\frac{89}{6} < w < \frac{103}{6}$.*

Proof: To determine the C^0-continuity of the 4-point quaternary scheme (4), we rewrite the polynomial (2) as

$$\gamma^{2,w}(z) = \left(\frac{z^3 + z^2 + z + 1}{4}\right)^0 b_0(z),$$

where

$$\begin{aligned} b_0(z) &= \left(\frac{z^3 + z^2 + z + 1}{4}\right)^3\Big[\Big(\frac{1}{6} + w\Big)z^6 + \Big(\frac{1}{2} - w\Big)z^5 + \Big(\frac{5}{6} - w\Big)z^4 \\ &\quad + \Big(1 + 2w\Big)z^3 + \Big(\frac{5}{6} - w\Big)z^2 + \Big(\frac{1}{2} - w\Big)z + \Big(\frac{1}{6} + w\Big)\Big]. \end{aligned}$$

The difference scheme of $b_0(z)$ which is $c_0(z)$ can be written as

$$\begin{aligned} c_0(z) &= \Big(\frac{1}{384} + \frac{w}{64}\Big)z^{12} + \Big(\frac{5}{384} + \frac{w}{64}\Big)z^{11} + \frac{7z^{10}}{192} + \Big(\frac{29}{384} + \frac{w}{64}\Big)z^9 \\ &\quad + \Big(\frac{47}{384} - \frac{w}{64}\Big)z^8 + \Big(\frac{31}{192} - \frac{w}{32}\Big)z^7 + \frac{17z^6}{96} + \Big(\frac{31}{192} - \frac{w}{32}\Big)z^5 \\ &\quad + \Big(\frac{47}{384} - \frac{w}{64}\Big)z^4 + \Big(\frac{29}{384} + \frac{w}{64}\Big)z^3 + \frac{7z^2}{192} + \Big(\frac{5}{384} + \frac{w}{64}\Big)z \\ &\quad + \Big(\frac{1}{384} + \frac{w}{64}\Big). \end{aligned}$$

Here $c_0(z)$ can be written as $c_0(z) = \sum_{i=0}^{12} r_0^i z^i$. We will check contractiveness of $c_0(z)$ by computing the infinity norm of $c_0(z)$

$$||c_0(z)||_\infty = \max\left\{\left|\sum_{i=0}^{3} |r_0^{4i}|, \sum_{i=0}^{2} |r_0^{4i+1}|, \sum_{i=0}^{2} |r_0^{4i+2}|, \sum_{i=0}^{2} |r_0^{4i+3}|\right|\right\}$$

so

$$\begin{aligned}||c_0(z)||_\infty &= \max\left\{\frac{|\frac{1}{6}+w|}{64}+\frac{|-\frac{47}{6}+w|}{32},\frac{|\frac{5}{6}+w|}{64}+\frac{|-\frac{31}{6}+w|}{32}+\frac{|\frac{29}{6}+w|}{64},\frac{1}{4},\right.\\ &\left.\frac{|\frac{5}{6}+w|}{64}+\frac{|-\frac{31}{6}+w|}{32}+\frac{|\frac{29}{6}+w|}{64}\right\}<1.\end{aligned}$$

It is straightforward to determine that $||c_0(z)||_\infty < 1$ for $-\frac{89}{6} < w < \frac{103}{6}$. This suggests that $c_0(z)$ is contractive, leading to the convergence of $b_0(z)$ and establishing the C^0-continuity of the scheme (4). This concludes the proof.
Likewise, we can determine the higher-order continuities of not only the 4-point and 5-point schemes but also other more schemes of high complexity in the family.

Theorem 3.2 *The 4-point quaternary subdivision scheme exhibits:*
- C^1*-continuous for* $-\frac{37}{6} < w < \frac{15}{2}$
- C^2*-continuous for* $-\frac{1}{6} < w < \frac{1}{2}$.

Theorem 3.3 *The 5-point quaternary subdivision scheme (5) exhibits:*
- C^0*-continuous for* $-\frac{721}{30} < w < \frac{163}{6}$
- C^1*-continuous for* $-\frac{295}{18} < w < \frac{397}{18}$
- C^2*-continuous for* $-\frac{41}{6} < w < \frac{55}{6}$
- C^3*-continuous for* $-5 < w < 3$.

4 Applications of the schemes

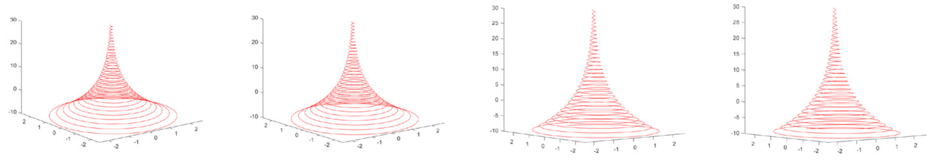
This section comprises two parts. The first part is dedicated to the applications of 4-point and 5-point schemes in curve fitting and modeling. The second part focuses on fitting and modeling regular surfaces, employing the 4-point and 5-point tensor product schemes. As the underlying schemes are quaternary, their product versions involve 16 rules for each scheme. The detailed mathematical expressions for these rules are extensive and not provided in this paper. They can be obtained straightforwardly through the tensor product of the curve-fitting rules for the respective schemes.

4.1 Curve fitting and modeling

In this subsection, data of various types is fitted and modeled using both schemes. This yields visually pleasing results.

Example-1: In this example, we generate an initial sketch based on the initial data obtained from a helical spring. It depicts a rough outline of a helical spring. Figure 2(a) illustrates the initial curve. Subsequently, Figures 2(b-c) showcase the curve fitted using the 4-point subdivision scheme (4) after two refinement steps for $w = -\frac{2}{6}$. The final smooth curve obtained after four iterations by the scheme is presented in Figure 2(d).

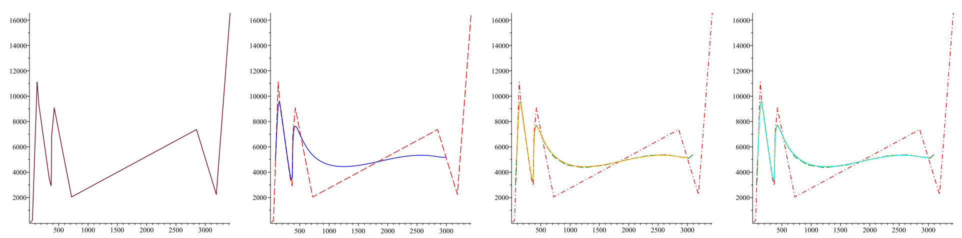
Example-2: In this example, we create an initial sketch using the initial data of pan-tumor canine cutaneous cancer histology patients. It was retrospectively selected from the biopsy archive of the Institute for Veterinary Pathology of the Freie University Berlin. In



(a) (b) (c) (d)

Figure 2. The curve generated using the subdivision scheme (4), utilizing a subdivision level of four and a smoothing parameter of $w = -\frac{2}{6}$. The initial sketch with its corresponding control points is depicted in part (a).

Figure 3(a), the brown curve represents the initial curve. Subsequently, in Figures 3(b-c), the curve is fitted using the subdivision scheme (5) after two refinement steps for $w = -2$. The final smooth green curve, fitted after four iterations by the 5-point scheme is depicted in Figure 3(d).



(a) (b) (c) (d)

Figure 3. The curve is fitted using the subdivision scheme (5) with $w = -2$ after four subdivision levels. In part (a), the initial sketch is presented along with the initial control points.

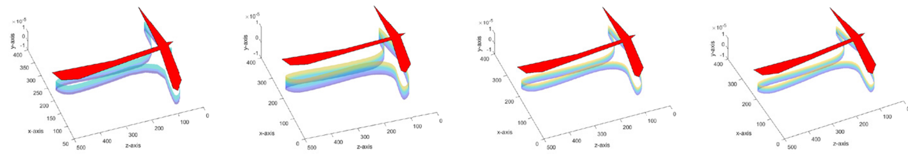
4.2 Surface fitting and images modeling

In this subsection, diverse regular shapes are generated using 4-point and 5-point quaternary schemes. These shapes are presented from various viewing angles. These shapes have pleasant and visually smooth characteristics.

Example-3: In this example, we extract data from the three-dimensional representation of the alphabet "T". The initial figure, depicted in Figure 4(a), serves as the starting point. Subsequent Figures 4(b-d) are the result of four iterations using the subdivision scheme (4) with a parameter value of $w = -\frac{1}{6}$.

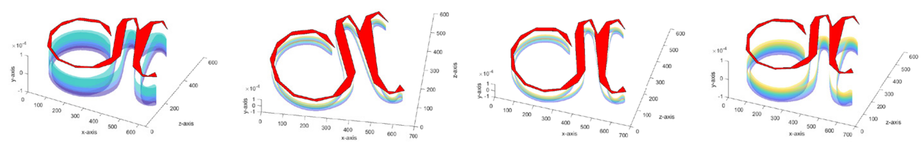
Example-4: In this example, we create an initial sketch using the three-dimensional data of the alphabet "n". Figure 5(a) represents the starting point which is the initial figure. Subsequent Figures 5(b-d) emerge after three iterations through the application of the subdivision scheme (4) with a parameter value of $w = \frac{1}{8}$.

Example-5: In this example, we create an initial sketch using the original data of the letter "E". Figures 6(b-c) depict the result after applying the scheme (5) two and three



(a) (b) (c) (d)

Figure 4. The surface shaped by the 4-point scheme (4) with $w = -\frac{1}{6}$, reaches its final form after undergoing four subdivision levels. Figure 4(a) showcases the initial sketch.

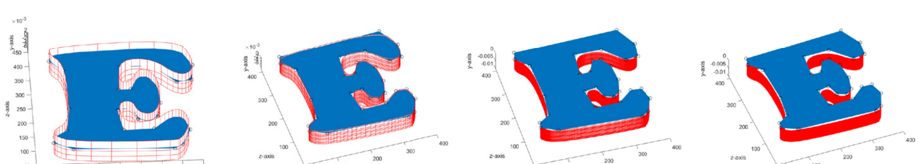


(a) (b) (c) (d)

Figure 5. The surface modeled by the 4-point scheme (4) with $w = \frac{1}{8}$, evolves through four subdivision levels. Figure 5(a) illustrates the starting point—an initial sketch.

times, respectively. Figure 6(d) showcases the final outcome-a smooth surface. It is achieved through the application of the scheme with a weight parameter of $w = 2$.

Example-6: In this example, we extract data from the three-dimensional representation of the alphabet "M". Figure 7(a) serves as the starting point-the initial figure. Subsequent figures are generated using the 5-point scheme at a parametric value of $w = 3$.

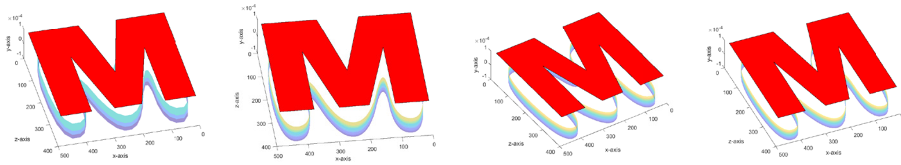


(a) (b) (c) (d)

Figure 6. The surface is shaped by the 5-point scheme (5) with $w = 2$ after four subdivision levels of the scheme. Figure 6(a) represents the initial sketch.

4.3 Conclusion

In this study, a new subdivision scheme is proposed to produce smooth curve and surface modeling images. The main significant contribution in this study is the highly accurate approximation and interpolation curves and surfaces. This will increase the resolution when we deal with the upscaling images especially for RGB type images. Future studies will be



(a) (b) (c) (d)

Figure 7. The surface is fitted by the 5-point scheme (5) with $w = 3$ after undergoing four subdivision levels of the scheme. Figure 7(a) depicts the initial sketch.

focusing on the application of the proposed subdivision scheme in RGB image interpolation such as fruit and medical imaging images [11, 12].

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