A 3-point point quinary approximating subdivision schemes and its application in geometric modeling and computer graphics

Abdul Ghaffar[1],*Ghulam* Mustafa[2] and *Samsul Ariffin* Abdul Karim[3,4,5,*]

[1]Department of Mathematics,Ghazi university D G Khan 32200, Pakistan
[2]Department of Mathematics, The Islamia University, Bahawalpur 63100, Pakistan
[3]Software Engineering Programme, Faculty of Computing and Informatics, Universiti Malaysia Sabah, Jalan UMS, 88400 Kota Kinabalu, Sabah, Malaysia
[4]Data Technologies and Applications (DaTA) Research Lab, Universiti Malaysia Sabah, Jalan UMS, 88400 Kota Kinabalu, Sabah, Malaysia
[5]Creative Advanced Machine Intelligence (CAMI) Research Centre, Universiti Malaysia Sabah, Jalan UMS, Kota Kinabalu 88400, Sabah, Malaysia

Abstract. This article discusses the significance of a subdivision scheme with shape parameters in geometric modeling and computational geometry. A new recursive method for generating the mask of 3-point quinary approximating subdivision schemes (ASSs) is presented. The proposed subdivision scheme exhibits all the geometric properties and offers better shape control than non-parametric subdivision schemes. The article includes several numerical examples to demonstrate the high practical value of the proposed schemes in geometric modeling and computer graphics.

1 Introduction

In the field of computational geometry and applied sciences, the use of Subdivision Surfaces (SSs) is limited due to their flaws. To reduce these deficiencies, a lot of work has been carried out by researchers [10–16]. One of the shortcomings of non-parametric SSs is their representation in polynomial form. Therefore, many researchers are searching for solutions to this problem in non-polynomial function space.

Over the past few decades, researchers have explored the use of trigonometric functions for describing curves and surfaces. These SSs are now playing an important role in several areas including medicine, electronics, and animations. Hao et al. [17] worked on the conditions for shape control parameters of the SS to obtain convexity and produce $C^r(r = 7, 8, 9)$ limit curves. Zhijie [18] proposed a shape preserving technique for classical interpolating 4-point SS. Wang and Li [19] derived the convexity preserving conditions for the ternary 4-point interpolatory SS in [20]. Albrecht and Romani, [21] discussed the convexity preserving conditions for the interpolatory SS with conic precision. Novara and Romani [22] presented the condition for assuring the convexity preservation of the 5-point interpolating SSs. Recently, computational geometry using SSs has received much attention. Mustafa et al. [23, 24] constructed parametric schemes based on a new kind of Lagrange polynomial

*e-mail: samsulariffin.karim@ums.edu.my,drsamsul.karim@gmail.com

with a single shape parameter. Ghaffar et al. [25, 26] formulated continuity conditions for the representation of free-form unified SSs with single shape control parameter. These free-form convoluted shapes can be attained by using shape control unified SSs. These newly defined approaches not only take over the advantages of non-parametric SSs but also resolve the issue of shape controlling of SSs with the help of tension control shape parameters. The proposed masks of the schemes provide different techniques to develop composite curves using higher smoothness conditions with straightforward and simple computation for proposed SSs because they are blended with linear polynomials rather than trigonometric functions.

In 2020, Ashraf et al. [27, 28] published a new method for the generalization of SSs with a single shape parameter. Shahzad et al. [29] constructed a new numerical algorithm to calculate the subdivision depth of binary SSs with shape parameters. Hussain et al. [30] established an algorithm that generates 5-point approximating SS of varying arity with a single shape parameter. To address these shortcomings, researchers have worked on techniques such as involving shape adjusting parameters in the subdivision process and using non-polynomial function spaces. Trigonometric functions have been used for the description of curves and surfaces, and have found applications in several fields. Recent proposed research has focused on constructing new 3-point ASSs with single shape parameter with C^2 continuity. This scheme have potential applications in computer graphics, computer animation, and multimedia technology. Limit curves and their properties are also discussed.

1.1 Construction of 3-point quinary ASSs

The family of (3)−points quinary ASS can be obtained from the sets of the coefficients (called the masks of the SSs and represented by $(\alpha^5_{3,t})$ of following polynomial

$$P^5_{3,t}(z) = (1 + z + z^2 + z^3 + z^4)^t \sum_{i=0}^{5(3)-4t-1} u_i z^i; \qquad t = 1, 2, 3, ...\frac{5n-3}{2}, \tag{1}$$

where $u_{5n-2t-3-i} = u_{5n-2t-3+i}$ and $u_{5n-2t-3} = \frac{1}{5^{t-1}} - 2\sum_{i=0}^{5n-2t-4} u_i$.

From (1), we obtained different polynomials $P^5_{3,t}(z)$ for each t. The superscript "5" of polynomials $P^5_{3,t}(z)$ and $\alpha^5_{3,t}(z)$ stands for quinary schemes while lower subscript "3" means (3)-point schemes. The shape parameter in the Eq. (1) gives us the opportunity to select the order of continuity for proposed scheme up to $C^{[\frac{5n-3}{2}]-1}$. For $n = 2$ in equation (1), we obtain,

$$P^5_{3,t}(z) = \left(1 + z + z^2 + z^3 + z^4\right)^t \sum_{i=0}^{15-4t-1} u_i z^i; \tag{2}$$

where $t = 1, 2, 3, ...\frac{5(2)-3}{2}$, $u_{7-2t-i} = u_{7-2t+i}$
and $u_{7-2t} = \frac{1}{5^{t-1}} - 2\sum_{i=0}^{6-2t} u_i$.

From the coefficient of the polynomial (2), we can generate family of 3-point quinary approximating schemes one scheme for each t.

For $t = 1$, we get

$$P^5_{3,1}(z) = \left(1 + z + z^2 + z^3 + z^4\right)^3 \sum_{ß=0}^{10} u_i z^i. \tag{3}$$

where $u_{5-i} = u_{5+i}$ and $i = 1, 2, 3, 4, 5,$
and $u_5 = 1 - 2(u_0 + u_1 + u_2 + u_3 + u_4).$
Then equation (3) becomes

$$\begin{aligned} P^5_{3,1}z &= \left(1 + z + z^2 + z^3 + z^4\right)\left(u_0 + u_1 z^1 + u_2 z^2\right. \\ &\quad +u_3 z^3 + u_4 z^4 + u_5 z^5 + u_6 z^6 + u_7 z^7 \\ &\quad \left.+u_8 z^8 + u_9 z^9 + u_{10} z^{10}\right). \end{aligned}$$

By simplification, we get

$$\begin{aligned} P^5_{3,1}(z) &= u_0 + (u_0 + u_1)z + (u_0 + u_1 + u_2)z^2 + (u_0 \\ &\quad +u_1 + u_2 + u_3)z^3 + (u_0 + u_1 + u_2 + u_3 \\ &\quad +u_4)z^4 + (u_1 + u_2 + u_3 + u_4 + u_5)z^5 \\ &\quad +(u_2 + u_3 + u_4 + u_5 + u_6)z^6 + (u_3 + u_4 \\ &\quad +u_5 + u_6 + u_7)z^7 + (u_4 + u_5 + u_6 + u_7 + u_8)z^8 \\ &\quad +(u_5 + u_6 + u_7 + u_8 + u_9)z^9 + (u_6 + u_7 + u_8 \\ &\quad +u_9 + u_{10})z^{10} + (u_7 + u_8 + u_9 + u_{10})z^{11} \\ &\quad +(u_8 + u_9 + u_{10})z^{12} + (u_9 + u_{10})z^{13} + u_{10} z^{14}. \end{aligned} \tag{4}$$

Substituting the values of u_5, u_6, u_7, u_8, u_9 and u_{10} in equation (4), we get

$$\begin{aligned} P^5_{3,1}(z) &= u_0 + (u_0 + u_1)z + (u_0 + u_1 + u_2)z^2 + (u_0 + u_1 \\ &\quad +u_2 + u_3)z^3 + (u_0 + u_1 + u_2 + u_3 + u_4)z^4 \\ &\quad +(1 - 2u_0 - u_1 - u_2 - u_3 - u_4)z^5 + (1 - 2u_0 - 2u_1 \\ &\quad -u_2 - u_3)z^6 + (1 - 2u_0 - 2u_1 - 2u_2)z^7 + \\ &\quad (1 - 2u_0 - 2u_1 - u_2 - u_3)z^8 + (1 - 2u_0 - u_1 \\ &\quad -u_2 - u_3 - u_4)z^9 + (u_0 + u_1 + u_2 + u_3 \\ &\quad +u_4)z^{10} + (u_0 + u_1 + u_2 + u_3)z^{11} \\ &\quad +(u_0 + u_1 + u_2)z^{12} + (u_0 + u_1)z^{13} + u_0 z^{14}. \end{aligned} \tag{5}$$

The coefficients (mask) of the above polynomial are

$$\begin{aligned} \alpha^5_{3,1} &= \{u_0, u_0 + u_1, u_0 + u_1 + u_2, u_0 + u_1 + u_2 \\ &\quad +u_3, u_0 + u_1 + u_2 + u_3 + u_4, 1 - 2u_0 - u_1 \\ &\quad -u_2 - u_3 - u_4 1 - 2u_0 - 2u_1 - u_2 - u_3, \\ &\quad 1 - 2u_0 - 2u_1 - 2u_2, 1 - 2u_0 - 2u_1 - u_2 - u_3, \\ &\quad 1 - 2u_0 - u_1 - u_2 - u_3 - u_4, u_0 + u_1 + u_2 + u_3 + u_4, \\ &\quad u_0 + u_1 + u_2 + u_3, u_0 + u_1 + u_2, u_0 + u_1, u_0\}. \end{aligned}$$

From above equation, we get

$$P^5_{3,2}(z) = (1 + z + z^2 + z^3 + z^4)^2 \sum_{i=0}^{6} u_i z^i, \tag{6}$$

where $u_{3-i} = u_{3+i}$ and $i = 1, 2, 3,$ and $u_3 = \frac{1}{5} - 2(u_0 + u_1 + u_2)$. Then

$$\begin{aligned} P^5_{3,2}z &= \left(1 + z + z^2 + z^3 + z^4\right)\left(u_0 + u_1 z^1 + u_2 z^2\right. \\ &\quad \left.+u_3 z^3 + u_4 z^4 + u_5 z^5 + u_6 z^6\right). \end{aligned}$$

By simplification, we get

$$
\begin{aligned}
P^5_{3,2}(z) \quad = \quad & \{u_0 + (2u_0 + u_1)z + (3u_0 + 2u_1 + u_2)z^2 \\
& +(4u_0 + 3u_1 + 2u_2 + u_3)z^3 \\
& +(5u_0 + 4u_1 + 3u_2 + 2u_3 + u_4)z^4 \\
& +(4u_0 + 5u_1 + 4u_2 + 3u_3 + 2u_4 + u_5)z^5 \\
& +(3u_0 + 4u_1 + 5u_2 + 4u_3 + 3u_4 + 2u_5 + u_6)z^6 \\
& +(2u_0 + 3u_1 + 4u_2 + 5u_3 + 4u_4 + 3u_5 \\
& +2u_6)z^7 + (u_0 + 2u_1 + 3u_2 + 4u_3 + 5u_4 \\
& +4u_5)z^8 + (u_1 + 2u_2 + 3u_3 + 4u_4 + 5u_5 \\
& +4u_6)z^9 + (u_2 + 2u_3 + 3u_4 + 4u_5 + 5u_6)z^{10} \\
& +(u_3 + 2u_4 + 3u_5 + 4u_6)z^{11} + (u_4 + 2u_5 + 3u_6)z^{12} \\
& +(u_5 + 2u_6)z^{13} + u_6 z^{14}\}.
\end{aligned}
$$

Substituting the values of u_3, u_4, u_5 and u_6, we get

$$
\begin{aligned}
P^5_{3,2}(z) \quad = \quad & \{u_0 + (2u_0 + u_1)\, z + (3u_0 + 2u_1 + u_2)\, z^2 \\
& + \left(\frac{1}{5} + 2u_0 + u_1\right) z^3 + \left(\frac{2}{5} + u_0\right) z^4 \\
& + \left(\frac{3}{5} - 2u_0\right) z^5 + \left(\frac{4}{5} - 4u_0 - 2u_1\right) z^6 \\
& + (1 - 6u_0 - 4u_1 - 2u_2)\, z^7 + \left(\frac{4}{5} - 4u_0 - 2u_1\right) z^8 \\
& + \left(\frac{3}{5} - 2u_0\right) z^9 + \left(\frac{2}{5} + u_0\right) z^{10} \\
& + \left(\frac{1}{5} + 2u_0 + u_1\right) z^{11} + (3u_0 + 2u_1 + u_2)\, z^{12} \\
& + (2u_0 + u_1)\, z^{13} + u_0 z^{14}\}.
\end{aligned} \tag{7}
$$

The coefficients (mask) of the above polynomial are

$$
\begin{aligned}
\alpha^5_{3,2} \quad = \quad & \{u_0, 2u_0 + u_1, 3u_0 + 2u_1 + u_2, \\
& \frac{1}{5} + 2u_0 + u_1, \frac{2}{5} + u_0, \frac{3}{5} - 2u_0, \\
& \frac{4}{5} - 4u_0 - 2u_1, 1 - 6u_0 - 4u_1 - 2u_2, \frac{4}{5} - 4u_0 - 2u_1, \\
& \frac{3}{5} - 2u_0, \frac{2}{5} + u_0, \frac{1}{5} + 2u_0 + u_1, 3u_0 \\
& +2u_1 + u_2, 2u_0 + u_1, u_0\}\,.
\end{aligned} \tag{8}
$$

Putting $n = 2$ and $t = 3$ in Eq. (1), we get

$$
P^5_{3,3}(z) = (1 + z + z^2 + z^3 + z^4)^3 \sum_{ß=0}^{2} u_i z^i. \tag{9}
$$

where $u_2 = u_0$. Then,

$$
P^5_{3,3}(z) = \left(1 + z + z^2 + z^3 + z^4\right)^3 \left(u_0 + u_1 z^1 + u_2 z^2\right). \tag{10}
$$

By simplification, we get

$$
\begin{aligned}
P^5_{3,3}(z) \quad = \quad & \{u_0 + (3u_0 + u_1)z + (6u_0 + 3u_1 + u_2)z^2 \\
& +(10u_0 + 6u_1 + u_2)z^3 + (15u_0 \\
& +10u_1 + 6u_2)z^4 + (18u_0 + 15u_1 + 10u_2)z^5 \\
& +(19u_0 + 18u_1 + 15u_2)z^6 + (18u_0 + 19u_1 \\
& +18u_2)z^7 + (15u_0 + 18u_1 + 19u_2)z^8 \\
& +(10u_0 + 15u_1 + 18u_2)z^9 + (6u_0 + 10u_1 \\
& +15u_2)z^{10} + (3u_0 + 6u_1 + 10u_2)z^{11} \\
& +(u_0 + 3u_1 + 6u_2)z^{12} + (u_1 + 3u_2)z^{13} \\
& +u_2 z^{14}\}. \qquad (11)
\end{aligned}
$$

Substituting the values of u_1 and u_2 in above equation, we get

$$
\begin{aligned}
P^5_{3,3}(z) \quad = \quad & \{u_0 + \left(u_0 + \frac{1}{25}\right)z + \left(u_0 + \frac{3}{25}\right)z^2 + \left(u_0 + \frac{6}{25}\right)z^3 \\
& + \left(u_0 + \frac{10}{25}\right)z^4 + \left(\frac{15}{25} - 2u_0\right)z^5 + \left(\frac{18}{25} - 2u_0\right)z^6 \\
& + \left(\frac{19}{25} - 2u_0\right)z^7 + \left(\frac{18}{25} - 2u_0\right)z^8 + \left(\frac{15}{25} - 2u_0\right)z^9 \\
& + \left(u_0 + \frac{10}{25}\right)z^{10} + \left(u_0 + \frac{6}{25}\right)z^{11} + \left(u_0 + \frac{3}{25}\right)z^{12} \\
& + \left(u_0 + \frac{1}{25}\right)z^{13} + u_0 z^{14}\}. \qquad (12)
\end{aligned}
$$

The coefficients (mask) of the above polynomial are

$$
\begin{aligned}
& \alpha^5_{3,3} = \Big\{u_0, u_0 + \tfrac{1}{25}, u_0 + \tfrac{3}{25}, u_0 + \tfrac{6}{25}, u_0 + \tfrac{10}{25}, \\
& \tfrac{15}{25} - 2u_0 \tfrac{18}{25} - 2u_0, \tfrac{19}{25} - 2u_0, \tfrac{18}{25} - 2u_0, \\
& \tfrac{15}{25} - 2u_0, u_0 + \tfrac{10}{25}, u_0 + \tfrac{6}{25}, u_0 + \tfrac{3}{25}, u_0 + \tfrac{1}{25}, u_0\Big\}.
\end{aligned} \qquad (13)
$$

2 Analysis of 3-point quinary scheme

Since the mask of the scheme (13) is given by

$$
\begin{aligned}
& \alpha^5_{(3,3)} = \Big\{u_0, u_0 + \tfrac{1}{25}, u_0 + \tfrac{3}{25}, u_0 + \tfrac{6}{25}, u_0 + \tfrac{10}{25}, \tfrac{15}{25} - 2u_0, \tfrac{18}{25} - 2u_0, \\
& \tfrac{19}{25} - 2u_0, \tfrac{18}{25} - 2u_0, \tfrac{15}{25} - 2u_0, u_0 + \tfrac{10}{25}, u_0 + \tfrac{6}{25}, \\
& u_0 + \tfrac{3}{25}, u_0 + \tfrac{1}{25}, u_0\Big\}.
\end{aligned} \qquad (14)
$$

The Laurent's Polynomial corresponding to this mask is given by

$$\begin{aligned} a(z) &= u_0 + \left(u_0 + \frac{1}{25}\right)z + \left(u_0 + \frac{3}{25}\right)z^2 + \left(u_0 + \frac{6}{25}\right)z^3 \\ &+ \left(u_0 + \frac{10}{25}\right)z^4 + \left(\frac{15}{25} - 2u_0\right)z^5 + \left(\frac{18}{25} - 2u_0\right)z^6 \\ &+ \left(\frac{19}{25} - 2u_0\right)z^7 + \left(\frac{18}{25} - 2u_0\right)z^8 + \left(\frac{15}{25} - 2u_0\right)z^9 \\ &+ \left(u_0 + \frac{10}{25}\right)z^{10} + \left(u_0 + \frac{6}{25}\right)z^{11} + \left(u_0 + \frac{3}{25}\right)z^{12} \\ &+ \left(u_0 + \frac{1}{25}\right)z^{13} + u_0 z^{14}. \end{aligned} \tag{15}$$

Now by using the formula

$$a^1(z) = \frac{5z^4}{(1 + z + z^2 + z^3 + z^4)} a(z).$$

This implies

$$\begin{aligned} a^1(z) &= 5z^4\{u_0 + \frac{1}{25}z + \frac{2}{25}z^2 + \frac{3}{25}z^3 + \frac{4}{25}z^4 + (\frac{5}{25} - 2u_0)z^5 \\ &+ \frac{4}{25}z^6 + \frac{3}{25}z^7 + \frac{2}{25}z^8 + \frac{1}{25}z^9 + u_0 z^{10}\}. \end{aligned}$$

Let S_1 be the SS corresponding to the Polynomial $a^1(z)$, then for C^0 continuity of scheme S, we require that $\|\frac{1}{5}S_1\|_\infty < 1$. The norm infinity of the scheme S_1 is given by

$$\begin{aligned} \left\|\frac{1}{5}S_1\right\|_\infty &= \frac{1}{5}\max\left\{\sum_j \left|b^{[1,1],}_{5j}\right|, \left|b^{[1,1],}_{5j+1}\right|,\right. \\ &\left.\left|b^{[1,1],}_{5j+2}\right|, \left|b^{[1,1],}_{5j+3}\right|, \left|b^{[1,1],}_{5j+4}\right|\right\}. \end{aligned}$$

This implies that

$$\begin{aligned} \left\|\frac{1}{5}S_1\right\|_\infty &= \frac{1}{5}\max\{(5|u_0| + |1 - 10u_0| + 5|u_0|), \\ &\left(\frac{5}{25} + \frac{20}{25}\right), \left(\frac{15}{25} + \frac{10}{25}\right), \\ &\left.\left(\frac{15}{25} + \frac{10}{25}\right), \left(\frac{5}{25} + \frac{20}{25}\right)\right\}. \end{aligned}$$

This implies

$$\left\|\frac{1}{5}S_1\right\|_\infty = \max\left\{\frac{1}{5}, 2|u_0| + \left|2u_0 - \frac{1}{5}\right|\right\}.$$

This implies that S_1 is contractive, and S is C^0 continuity if and only if $\frac{-1}{5} < u_0 < \frac{3}{10}$.
By using

$$a^2(z) = \frac{5z^4}{(1 + z + z^2 + z^3 + z^4)} a^1(z).$$

This implies

$$a^2(z) = 25z^8 \left\{u_0 + \left(\frac{1}{25} - u_0\right)z + \frac{1}{25}z^2 + \frac{1}{25}z^3 + \frac{1}{25}z^4 + \left(\frac{1}{25} - u_0\right)z^5 + u_0 z^6\right\}.$$

Let S_2 be the SS corresponding to the Polynomial $a^1(z)$, then for C^0 continuity of scheme S, we require that $\|\frac{1}{5}S_2\|_\infty < 1$.
The norm infinity of the scheme S_2 is given by

$$\begin{aligned}\left\|\frac{1}{5}S_2\right\|_\infty &= \frac{1}{5}\max\{(25\,|u_0| + |1 - 25u_0|),\\ &\quad (25\,|u_0| + |1 - 25u_0|), 1, 1, 1\}.\end{aligned}$$

This implies

$$\left\|\frac{1}{5}S_2\right\|_\infty = \max\left\{\frac{1}{5}, 5\,|u_0| + \left|5u_0 - \frac{1}{5}\right|\right\}.$$

This implies that S_2 is contractive, and S is C^1 continuity if and only if $\frac{-2}{25} < u_0 < \frac{3}{25}$.
Now by using (**??**), (**??**) and (15), we get

$$a^3(z) = \frac{5z^4}{(1 + z + z^2 + z^3 + z^4)}a^2(z).$$

This implies

$$a^3(z) = 125z^{12}\{u_0 + \left(\frac{1}{25} - 2u_0\right)z + u_0 z^2\}.$$

Let S_3 be the SS corresponding to the Polynomial $a^1(z)$, then for C^0 continuity of scheme S, we require that $\|\frac{1}{5}S_3\|_\infty < 1$. The norm infinity of the scheme S_3 is given by

$$\left\|\frac{1}{5}S_2\right\|_\infty = \frac{1}{5}\max\{(125\,|u_0| + |5 - 250u_0|), (125\,|u_0|)0, 0\}.$$

This implies

$$\left\|\frac{1}{5}S_2\right\|_\infty = \max\{25\,|u_0|\,, |50u_0 - 1|\}.$$

This implies that S_3 is Contractive, and S is C^2 continuous if and only if $0 < u_0 < \frac{1}{25}$.
No further continuity of the scheme exists.

2.1 Support of scheme

The mask of the 3-point quinary scheme has 15 terms. The middle value of our this mask is situated at the distance of 7 term from the left and right extreme non-zero coefficients. When we perform the first refinement level this distance reduces by $\frac{7}{5}$ from each side. In this way, at every refinement step the sequence of the mask will continue to decrease by the same factor $\frac{7}{5} \times \frac{1}{5}$. Consequently, after k subdivision levels, the distance of the extreme terms from the central term can be calculated by the formula,

$$7\left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^2} +, \ldots, \frac{1}{5^k}\right) = \frac{7}{5}\left(\sum_{j=0}^{k-1}\frac{1}{5^k}\right).$$

So, when $k \to \infty$, the total width of support of the limiting curve will be given by,

$$\frac{7}{5}\left(\frac{7}{5}\sum_{j=0}^{k-1}\frac{1}{5^k}\right) + \frac{7}{5}\left(\frac{7}{5}\sum_{j=0}^{k-1}\frac{1}{5^k}\right) = \frac{14}{5}\left(\sum_{j=0}^{k-1}\frac{1}{5^k}\right) = 3.5.$$

Graphical representation of support is showed in Figure 1.

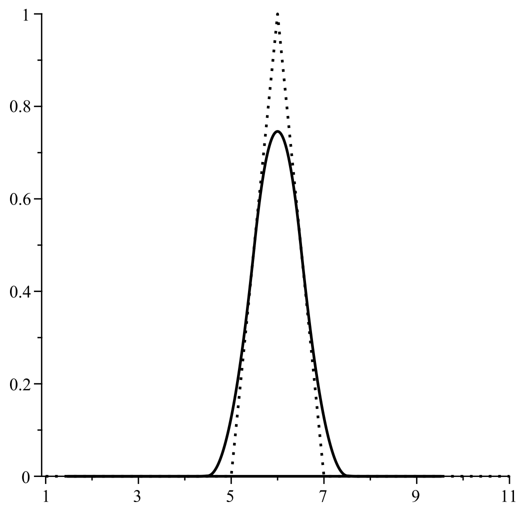


Figure 1. *(3-point quinary scheme.)*

3 Error bounds:

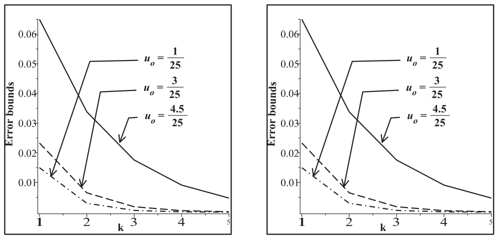
From the research conducted in Table 1 using [32] with $\chi = 0.1$, we have calculated error bounds between the limit curve and the control polygon after k-fold subdivision of quinary approximating schemes. Our analysis of Figure. 2 and Tables 1 has led us to the following conclusion: The error bounds of these schemes increase as the complexity (number of points involved to insert new points) of the SS increases. However, in the case of 3-point quinary approximating schemes, the error bounds remain constant and achieve their minimum value over $0 \leq u_0 \leq \frac{1}{10}$. It is evident that the error bound of quinary approximating schemes gradually increases on both sides of the interval.

Table 1. Error bounds for 3-point quinary ASS:

Scheme	Parameter	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
3-point	$u_0 = \frac{9}{50}$	0.065000	0.033800	0.017576	0.009140	0.004753
...	$u_0 = \frac{3}{25}$	0.033750	0.012150	0.004374	0.001575	0.000567
...	$0 \leq u_0 \leq \frac{1}{10}$	0.015000	0.003000	0.000600	0.000120	0.000024
...	$u_0 = -\frac{1}{25}$	0.0023333	0.006533	0.001829	0.000512	0.000143
...	$u_0 = -\frac{2}{25}$	0.065000	0.033800	0.017576	0.009140	0.004753

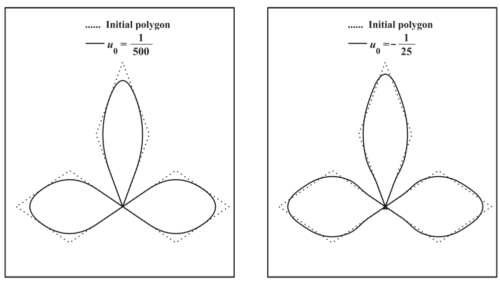
4 Geometrical significance and visual inspection

The presented schemes produce visually smooth curves that can be customized by altering the shape parameter within their domain. By adjusting this parameter, we can achieve remarkable beautification and attraction in the figures. For instance, Figures 3 showcase the graphs for 3-point quinary approximating SSs with varying shape parameter values. We can easily obtain continuous curves that satisfy smoothness conditions and modify the subdivision curve as desired by adjusting the shape parameter, as exemplified in Figures 3.

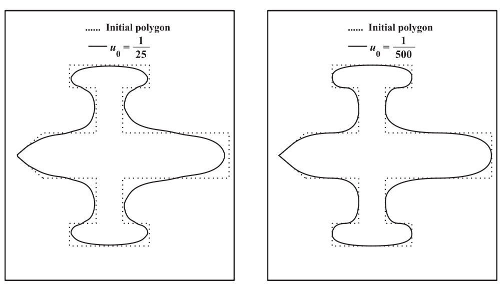


a) (b)

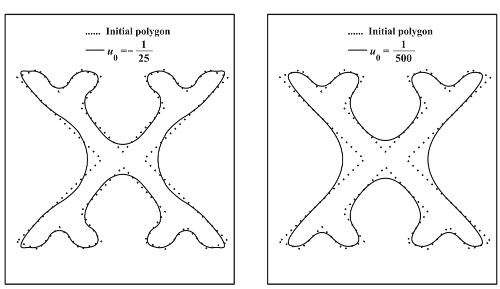
Figure 2. *(a) and (b) represent the variation of error bounds of 3-point quinary scheme*



(a) (b)



(c) (d)



(e) (f)

Figure 3. *(a)-(f) gives parameter's justification of 3-point quinary scheme.*

5 Conclusion

In this section, we have analyzed the 3-point quinary approximating schemes and discussed their properties. We used the Laurent polynomial method to analyze the smoothness of the scheme. We also compared them with some existing schemes. Several examples are provided to illustrate that the proposed schemes give geometric designers a wide choice to generate smooth geometric models as per their needs. In terms of application in computer graphics, the resulting curves are smooth and highly suitable in designing any computer graphics shape and model. One possible future extension is the generalization of the proposed subdivision scheme in scattered data interpolation and image interpolation problems [33, 34].

Acknowledgement

This research was fully supported by Ministry of Higher Education (MOHE) of Malaysia through Fundamental Research Grant Scheme **[FRGS/1/2023/ICT06/ UMS/02/1] (New Scattered Data Interpolation Scheme Using Quasi Cubic Triangular Patches for RGB Image Interpolation)** and Universiti Malaysia Sabah. Special thanks to the Faculty of Computing and Informatics, Universiti Malaysia Sabah for the tremendous computing facilities support.

References

[1] Rham, G. D. (1956). Sur une courbe plane. J. Math. Pures Appl.(9), 35, 25-42.

[2] Chaikin, G. M. (1974). An algorithm for high-speed curve generation. Computer graphics and image processing, 3(4), 346-349.

[3] Riesenfeld, R. F. (1975). On Chaikin's algorithm. Computer Graphics and Image Processing, 4(3), 304-310.

[4] Catmull, E., & Clark, J. (1978). Recursively generated B-spline surfaces on arbitrary topological meshes. Computer-aided design, 10(6), 350-355.

[5] Doo, D., & Sabin, M. (1978). Behaviour of recursive division surfaces near extraordinary points. Computer-Aided Design, 10(6), 356-360.

[6] Ball, A. A., & Storry, D. J. T. (1986). A matrix approach to the analysis of recursively generated B-spline surfaces. Computer-Aided Design, 18(8), 437-442.

[7] Dyn, N., Levin, D., & Gregory, J. A. (1987). A 4-point interpolatory subdivision scheme for curve design. Computer aided geometric design, 4(4), 257-268.

[8] Dyn, N., Levine, D., & Gregory, J. A. (1990). A butterfly subdivision scheme for surface interpolation with tension control. ACM transactions on Graphics (TOG), 9(2), 160-169.

[9] Reif, U. (1995). A unified approach to subdivision algorithms near extraordinary vertices. Computer Aided Geometric Design, 12(2), 153-174.

[10] Dyn, N.; Iske, A.; Quak, E.; Floater, M.S. Tutorials on Multiresolution in Geometric Modelling, Summer School Lecture Notes Series: *Mathematics and Visualization; Springer Science & Business Media: Berlin, Germany*, *2002*.

[11] Mustafa, G.; Hashmi, M.S. Subdivision depth computation for *n*-ary subdivision curves/surfaces. *Vis. Comput.* **2010**, 26, 841-851.

[12] Shang, Y. Lack of Gromov-hyperbolicity in small-world networks. *Cent. Eur. J. Math.* **2011**, 10, 1152-1158.

[13] Siddiqi, S.S.; Younis, M. The Quaternary Interpolating Scheme for Geometric Design. *Int. Sch. Res. Not.* **2013**, 2013.

[14] Mustafa, G.; Ashraf, P.; Deng, J. Generalized and unified families of interpolating subdivision schemes. *Numer. Math. Theory Method. Appl.* **2014**, 7, 193-213.

[15] Rehan, K.; Siddiqi, S.S. A Family of Ternary Subdivision Schemes for Curves. *Appl. Math. Comput.* **2015**, 270, 114-123.

[16] Rehan, K.; Sabri, M.A. A combined ternary 4-point subdivision scheme. *Appl. Math. Comput.* **2016**, 276, 278-283.

[17] Y.-X. Hao, R.-H. Wang, C.-J. Li, Analysis of a 6-point binary subdivision scheme, Applied Mathematics and Computation 218 (7) (2011) 3209–3216.

[18] C. Zhijie, Four-point scheme and convexy-preserving algorithm, Journal of Computer-Aided Design and Computer Graphics.

[19] Y. Wang, Z. Li, A family of convexity-preserving subdivision schemes, Journal of Mathematical Research with Applications 37 (4) (2017) 489–495.

[20] M. F. Hassan, I. Ivrissimitzis, N. A. Dodgson, M. A. Sabin, An interpolating 4-point c2 ternary stationary subdivision scheme, Computer Aided Geometric Design 19 (1) (2002) 1–18.

[21] G. Albrecht, L. Romani, Convexity preserving interpolatory subdivision with conic precision, Applied Mathematics and Computation 219 (8) (2012) 4049–4066.

[22] P. Novara, L. Romani, On the interpolating 5-point ternary subdivision scheme: A revised proof of convexity-preservation and an application-oriented extension 147.

[23] Mustafa, G., Khan, F., & Ghaffar, A. (2009). The *m*-point approximating subdivision scheme. Lobachevskii Journal of Mathematics, 30(2), 138-145.

[24] Mustafa, G., Ghaffar, A., & Khan, F. (2011). The odd-point ternary approximating schemes. American Journal of Computational Mathematics, 1(02), 111.

[25] Ghaffar, A.; Mustafa, G.; Qin, K. Unification and application of 3-point approximating subdivision schemes of varying arity. *Open J. Appl. Sci.* **2012**, 2, 48-52.

[26] Ghaffar, A.; Mustafa, G.; Qin, K. The 4-point 3-ary approximating subdivision scheme. *Open J. Appl. Sci.* **2013**, 3, 106-111.

[27] Ashraf, P., Nawaz, B., Baleanu, D., Sooppy Nisar, K., Ghaffar, A., Khan, M. A. A., & Akram, S. (2020). Analysis of geometric properties of ternary four-point rational interpolating subdivision scheme. Mathematics, 8(3), 338.

[28] Ashraf, P., Ghaffar, A., Baleanu, D., Sehar, I., Nisar, K. S., & Khan, F. (2020). Shape-Preserving Properties of a Relaxed Four-Point Interpolating Subdivision Scheme. Mathematics, 8(5), 806.

[29] Shahzad, A.; Khan, F.; Ghaffar, A.; Mustafa, G.; Nisar, K.S.; Baleanu, D. A novel numerical algorithm to estimate the subdivision depth of binary subdivision schemes. *Symmetry* **2020**, 12, 66.

[30] Hussain, S. M., Rehman, A. U., Baleanu, D., Nisar, K. S., Ghaffar, A., & Abdul Karim, S. A. (2020). Generalized 5-point approximating subdivision scheme of varying arity. Mathematics, 8(4), 474.

[31] N.Dyn, M. S. Floater and K.Horman, A C^2 Four-Point Subdivision Scheme with Fourth Order Accuracy and its Extension, in Mathematical Methods for Curves and Surfaces: Tromso 2004, M. Daehlen, K. Morken, and L. L. Schumaker (eds.), 2005, 145-156.

[32] Mustafa, G., & Hashmi, M. S. (2010). Subdivision depth computation for n-ary subdivision curves/surfaces. The Visual Computer, 26(6-8), 841-851.

[33] Karim, S.A.A. Rational bi-quartic spline with six parameters for surface interpolation with application in image enlargement. *IEEE Access* **2020**, *8*, 115621–115633.

https://doi.org/10.1109/ACCESS.2020.3002387.

[34] Karim, S.A.A.; Saaban, A.; Nguyen, V.T. Scattered data interpolation using quartic triangular patch for shape-preserving interpolation and comparison with mesh-free methods. *Symmetry* **2020**, *12*, 1071. https://doi.org/10.3390/sym12071071.