Investigation of the fixed-point theorem on a complete cone metric space

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Abstract. Metric space is the one of topics in mathematical analysis which is still being studied and developed. Previous research has extended the contraction mappings and completeness properties from metric spaces to other spaces such as cone metric spaces. This research will prove a fixed-point theorem on cone metric spaces, construct a function and maps to [0,1), then add several conditions to the theorem, namely partially ordered, contraction mapping, and non-decreasing and continuous functions. The outcomes include evidence indicating a unique fixed point in cone metric space. As consequence, there are three different characteristics for contraction mapping expressed in three corollaries.

1 Introduction

Functional analysis is an important part of mathematical analysis discussing the concept of space accompanied by certain functions that are defined in that space. Types of spaces discussed include metric, normed, Banach, inner product, and Hilbert spaces. This is defined as a non-empty set accompanied by a distance function that applies the concept of the convergence of sequences [4]. One of theorems that utilizes the conviction of convergence in metric spaces is the Banach's fixed-point theorem.

Stefan Banach demonstrated his fixed-point theorem within a comprehensive metric space [4]. This is acknowledged as the Banach's contraction principle. This theorem stated that the fixed point of the contraction mapping in a complete metric space, there is a sole fixed point that exists. A contraction mapping is a function that diminishes the separation among the points' images compared relative to the point-to-point distance in space. A fixed point of a function is defined by a fixed point is a point that maps to itself. In a complete metric space, fixed points can be defined. A metric space is considered complete if every Cauchy sequence within it converges.

Mathematicians have applied the general idea of Banach's fixed point theorem to expanded metric spaces. In 2006, Guang and Xian extended the concept of metric spaces to include cone metric spaces, derived fixed-point theorems for mappings satisfying specific diminishing conditions, laying the groundwork for research in cone metric spaces [3]. The other studies related to cone metric spaces were carried out by [1], [2], and [5]. First, Bahtiar, et al have concluded that convergence sequences and Cauchy sequences in metric spaces also hold in cone metric spaces. It is also obtained that the diminishing mapping on the cone

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metric space with a normal cone exists a unique fixed point. Then, they demonstrated that the extension of the contraction mappings to the complete cone metric space also has a unique fixed point. Furthermore, Zuhra, et al proved the principle of contraction developed into further elaboration by changing a constant into a function. Differ with Guang and Xian that used the principle of contraction with a constant at a certain interval, this research will replace the constant with a function from a certain interval to a set of real numbers.

2 Preliminaries

The subsequent are definitions, lemmas, and theorems in cone metric spaces that will be used in this investigation.

Definition 1. [3] If X is a real Banach spaces and C is a subset of X. C is a cone if and only if:

(a) C contains at least one element and closed and C has a non-zero element;
(b) If α, β are non-negative elements in $\mathbb{R}$, and u, v are elements in C then the linear combination $\alpha u + \beta v$ is also in C;
(c) If u is in C and $(-u)$ is also in C then u equals zero.

We can see from the definition of a cone that the additive invers of non-zero elements in a cone does not exist.

Definition 2. [4] Let Q be a set that is equipped with a partial ordering, which is a binary relation $\preccurlyeq$ that satisfies the conditions:

(a) $q \preccurlyeq q$, for all $q \in Q$ (This condition is called Reflexivity);
(b) if $q_1 \preccurlyeq q_2$ and $q_2 \preccurlyeq q_1$, then $q_1 = q_2$ (This condition is called Anti-symmetry);
(c) if $q_1 \preccurlyeq q_2$ and $q_2 \preccurlyeq q_3$, then $q_1 \preccurlyeq q_3$ (This condition is called Transitivity),

then Q is called a partially ordered set.

Definition 3. [3] Partially ordered properties in a cone $C \subseteq X$ satisfy the following conditions,

(a) for all a, b in X, $a \preccurlyeq b$ if and only if $b - a$ is in C;
(b) for all a, b in $X, a \prec b$ if and only if $b - a$ is in C and $b \neq a$;
(c) $a \preccurlyeq b$ if and only if $b - a$ is in $int(C)$.

Definition 4. [2] For any nonempty set S, the space $(S, d, \preccurlyeq)$ is a partially ordered metric space if the pair $(S, \preccurlyeq)$ forms a partially ordered set; and (S, d) is a metric space.

Definition 5. [3] Suppose S constitute a set, X be a Banach space. Consider the mapping $d_c: S \times S \to X$ satisfies :

(a) $0 \preccurlyeq d_c(s, t)$, for all s, t in S dan $d_c(s, t) = 0$ if and only if $s = t$;
(b) $d_c(s, t) = d_c(t, s)$, for all s, t in S;
(c) $d_c(s, t) \preccurlyeq d_c(s, u) + d_c(u, t)$, for all s, t, u in S.

Then d_c is a cone metric on S and (S, d_c) is a cone metric space.
An example of a cone metric space is given below. This example should be familiar to the readers.

Example 1 : Let $X = \mathbb{R}^2$, $C = \{(c_1, c_2) \in X | c_1, c_2 \geq 0\} \subset \mathbb{R}^2$ and suppose $S = \mathbb{R}^2$ and d_c is defined as follow. If $s = (s_1, s_2)$ and $t = (t_1, t_2)$ then

$$d_c(s,t) = (|s_1 - t_1|, \alpha|s_2 - t_2|)$$

where $\alpha \geq 0$ is a constant. Then (C, d_c) is a cone metric

Definition 6. [3] Suppose (C, d_c) be a cone metric space, (c_n) be a sequence in C and c is an element in C. A sequence (c_n) is to be convergent to c if and only if for all element x in X with $0 \preccurlyeq x$, there exist $n_0 \in \mathbb{N}$, such that for all $n > n_0$ implies $d_c(c_n, c) \preccurlyeq x$. We denoted this by $\lim_{n\to\infty.} c_n = c$ or $c_n \to c$, for $n \to \infty$.

Example 2 : Let $S = \mathbb{R}, X = \mathbb{R}$ and $C = \{c \in \mathbb{R} | c \geq 0\}$ with a cone metric $d_c: S \times S \to X$ with $d_c(s,t) = |s - t|$. A sequence (s_n) in S with $s_n = \frac{1}{n}$ satisfies $\lim_{n\to\infty.} s_n = 0$.

Definition 7. [3] Let (S, d_c) be a cone metric space. Let (s_n) form a sequence in S. If for any element x in X with $0 \preccurlyeq x$, there exist a positive integer *N* such that for all $k, n > N$ implies $d_c(s_k, s_n) \preccurlyeq x$, then (s_n) is referred to as a Cauchy sequence in *X*.

Definition 8. [3] In a cone metric space (S, d_c) assuming each Cauchy sequence converges in *X*, then *X* is termed a complete cone metric space.

Lemma 1. [3] Consider (C, d_c) as a cone metric space, and let (c_n) be a sequence in *C*. If (c_n) converges to *c*, then (c_n) forms a Cauchy sequence.

Now, the following theorem presented is the main result of reference [3].

Theorem 5. [3] Assume (S, d_c) forms a complete cone metric space, *C* represents a normal cone with a normal constant *m*. Let $g: S \to S$ fulfill the contraction condition.

$$d_c\big(g(s), g(t)\big) \preccurlyeq m d_c(s,t), \text{ for all } s,t \text{ in } S,$$

where *m* in the interval [0, 1), then g possesses a unique fixed point in *S*. Additionally, for any element *s* in *S*, the iterative sequence $(g^n(s))$ converges to this fixed point.

Then, Zuhra, et al has proved their main results as follows.

Theorem 6. [5] Suppose *X* represent a real Banach space, with *C* as a cone in *X*. Regard $(S, d, \preccurlyeq)$ be a complete partially ordered cone metric space with *c*-distance *q*. The mappings $g: S \to S$ and $m: S \to [0,1)$ adhere to the subsequent conditions:

(1) There is an element s_0 in S such that $s_0 \preccurlyeq g(s_0)$;
(2) For all element s in S, the inequality $m(g(s)) \leq m(s)$ holds;
(3) There exists u where u is an element in $M_{s,t}$, that is

$$M_{s,t} \equiv \left\{q(s,t), q\big(s, g(s)\big), q\big(t, g(t)\big), \frac{1}{2}\Big(q(s, g(t)) - q\big(t, g(s)\big)\Big)\right\}$$

such that $d\big(g(s), g(t)\big) \leq m(s)u$ for all s, t in S with $t \preccurlyeq s$; such that $d\big(g(s), g(t)\big) \leq m(s)u$ for all s, t in S with $t \preccurlyeq s$;

(4) g is continuous and nondecreasing.

Then there is a fixed point of g in S. Moreover, if $g(v) = v$, then $q(v, v) = 0$.

3 Main results

Now, we will prove the main theorem of this paper.

Theorem. Suppose $(S, d_c, \preccurlyeq)$ be a partially ordered cone metric space and there exists maps $g: S \to S$ and $m: S \to [0,1)$ that fulfill the following conditions :

(1) There is an element s_0 in S such that $s_0 \preccurlyeq g(s_0)$;
(2) $m(g(s)) \leq m(s)$ for all $s \in S$;
(3) $d_c\big(g(s), g(t)\big) \leq m(s) d_c(s,t)$ for all s, t in S with $t \preccurlyeq s$;
(4) g is continuous and nondecreasing.

Then there is a fixed point of g in S. Moreover, if $g(v) = v$, then $d_c(v, v) = 0$.

Proof: The proof is partitioned into four distinct steps

1. We will show the existence of partially ordered elements.
If $g(s_0) = s_0$, then s_0 is a fixed point of g and the theorem is proved. Surmise $g(s_0) \neq s_0$. Choose $s_0 \in S$, we set

$$\begin{aligned}
&s_1 = g(s_0),\\
&s_2 = g(s_1) = g(g(s_0)) = g^2(s_0),\\
&s_3 = g(s_2) = g\big(g(s_1)\big) = g\Big(g\big(g(s_0)\big)\Big) = g^3(s_0),\\
&\vdots\\
&s_n = g(s_{n-1}) = g\big(g(s_{n-2})\big) = \cdots = g^n(s_0),\\
&s_{n+1} = g(x_n) = g(g(x_{n-1})) = \cdots = g^{n+1}(s_0).
\end{aligned}$$

Since $s_0 \preccurlyeq g(s_0)$ and g is nondecreasing then, by induction, we have $s_0 \preccurlyeq g(s_0) \preccurlyeq g^2(s_0) \preccurlyeq \cdots \preccurlyeq g^n(s_0) \preccurlyeq g^{n+1}(s_0) \preccurlyeq \cdots$. Or, in other words, we get a non-decreasing sequence (s_n) in S.

2. Our aim is to demonstrate $d_c(s_n, s_{n+1}) \leq m^n(s_0) d_c(s_0, s_1)$.
Claim that $d_c(s_n, s_{n+1}) \leq m(s_{n-1}) d_c(s_{n-1}, s_n), \quad n \geq 1$ (3.1)
By condition (3) dan $s_n = g(s_{n-1})$, we obtain

$$\begin{aligned}
d_c(s_n, s_{n+1}) &= d_c\big(g(s_{n-1}), g(s_n)\big)\\
&\leq m(s_{n-1}) d_c(s_{n-1}, s_n). \qquad n \geq 1
\end{aligned}$$

Then by inequality (3.1),

$$\begin{aligned}
d_c(s_n, s_{n+1}) &\leq m(s_{n-1}) d_c(s_{n-1}, s_n),\\
&= m\big(g(s_{n-2})\big) d_c(s_{n-1}, s_n),\\
&\leq m(s_{n-2}) d_c(s_{n-1}, s_n.
\end{aligned}$$

Continuing the process by using the fact that $s_k = g(s_{k-1})$ for any $k = 1, \dots, n$ and applying condition (2) then we have

$$d_c(s_n, s_{n+1}) \leq m(s_0) d_c(s_{n-1}, s_n).$$

Since (s_n) is non-decreasing and by applying condition (3), we have

$$d_c(s_n, s_{n+1}) \leq m(s_0) d_c(s_{n-1}, s_n) \leq m(s_0) m(s_{n-2}) d_c(s_{n-2}, s_{n-1}).$$

Continuing applying condition (3), we have

$$\begin{aligned} d_c(s_n, s_{n+1}) &\leq m(s_0)m(s_0)d_c(s_{n-2}, s_{n-1}), \\ &= m^2(s_0)d_c(s_{n-2}, s_{n-1}), \\ &\leq m^2(s_0)m(s_0)d_c(s_{n-3}, s_{n-2}), \\ &= m^3(s_0)d_c(s_{n-3}, s_{n-2}), \\ &\leq m^3(s_0)m(s_0)d_c(s_{n-4}, s_{n-3}), \\ &= m^4(s_0)d_c(s_{n-4}, s_{n-3}), \\ &\ \vdots \\ d_c(s_n, s_{n+1}) &\leq m^n(s_0)d_c(s_0, s_1). \end{aligned} \tag{3.2}$$

3. Next, we display that (s_n) is a Cauchy sequence.
   Take $k > n \geq 1$. Then,

$$\begin{aligned} d_c(s_n, s_k) &\leq d_c(s_n, s_{n+1}) + d_c(s_{n+1}, s_k), \\ &\leq d_c(s_n, s_{n+1}) + d_c(s_{n+1}, s_{n+2}) + d_c(s_{n+2}, s_m), \\ &\leq d_c(s_n, s_{n+1}) + d_c(s_{n+1}, s_{n+2}) + \cdots + d_c(s_{k-2}, s_{k-1}) + \\ &\quad d_c(s_{k-1}, s_k). \end{aligned} \tag{3.3}$$

   Using inequalities (3.2) dan (3.3), we gain

$$\begin{aligned} d_c(s_n, s_k) &\leq d_c(s_n, s_{n+1}) + d_c(s_{n+1}, s_{n+2}) + \cdots + d_c(s_{k-2}, s_{k-1}) + \\ &\quad d_c(s_{k-1}, s_k), \\ &\leq m^n(s_0)d_c(s_0, s_1) + m^{n+1}(s_0)d_c(s_0, s_1) + \cdots + \\ &\quad m^{k-2}(s_0)d_c(s_0, s_1) + m^{k-1}(s_0)d_c(s_0, s_1), \\ &\leq [m^n(s_0) + m^{n+1}(s_0) + \cdots + m^{k-2}(s_0) + m^{k-1}(s_0)]d_c(s_0, s_1), \\ &\leq \frac{m^n(s_0)}{1 - m(s_0)} d_c(s_0, s_1). \end{aligned}$$

   Since $m(s_0) \in [0,1)$, then we have $\frac{m^n(s_0)}{1-m(s_0)} d_c(s_0, s_1) \to 0$ for $n \to \infty$.
   So that (s_n) is a Cauchy sequence.

4. In this last step, we will show the existence of the fixed point.
   As S is complete, there exist $s^* \in S$ such that $s_n \to s^*$ for $n \to \infty$.
   Since g is continuous, we get

$$s^* = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} g(s_n) = g\left(\lim_{n\to\infty} s_n\right) = g(s^*).$$

   This shows that s^* is a fixed point of g.
   Now, suppose that t^* is another fixed point of g. Then

$$\begin{aligned} d_c(s^*, t^*) = d_c\big(g(s^*), g(t^*)\big) &\leq m(s^*)d_c(s^*, t^*), \\ &\leq m(s^*)d_c(s^*, t^*). \end{aligned}$$

   Or,

$$\begin{aligned} &d_c(s^*, t^*) - m(s^*)d_c(s^*, t^*) \leq 0\ , \\ &\big(1 - m(s^*)\big)d_c(s^*, t^*) \leq 0. \end{aligned}$$

Since $m(s^*) \in [0,1)$, then $m(s^*) = 0$. Hence, $d_c(s^*, t^*) = 0$. So, $s^* = t^*$.
We have proved that g only has a unique fixed point s^*.

This theorem yields the following corollaries in cone metric spaces. The difference between these 3 corollaries is only in condition 3 of the theorem.

Corollary 1. Let $(S, d_c, \preccurlyeq)$ be cone metric space that is partially ordered and complete. There exist mappings $g: S \to S$ and $m: S \to [0,1)$ such that the following hold:

(1) There is an element s_0 in S such that $s_0 \preccurlyeq g(s_0)$;
(2) $m(g(s)) \le m(s)$ for all $s \in S$;
(3) $d_c\big(g(s), g(t)\big) \le m(s) d_c(s, g(s))$ for all $s, t \in S$ with $t \preccurlyeq s$;
(4) g is continuous and nondecreasing.

Then there is a fixed point of g in S. Moreover, if $g(v) = v$, then $d_c(v, v) = 0$.

Corollary 2. Let $(S, d_c, \preccurlyeq)$ be cone metric space that is partially ordered and complete. There exists maps $g: S \to S$ and $m: S \to [0,1)$ that fulfill the following conditions:

(1) There is an element s_0 in S such that $s_0 \preccurlyeq g(s_0)$;
(2) $m(g(s)) \le m(s)$ for all $s \in S$;
(3) $d_c\big(g(s), g(t)\big) \le m(s) d_c(t, g(t))$ for all $s, t \in S$ with $t \preccurlyeq s$;
(4) g is continuous and nondecreasing.

Then there is a fixed point of g in S. Moreover, if $g(v) = v$, then $d_c(v, v) = 0$.

Corollary 3. Let $(S, d_c, \preccurlyeq)$ be a cone metric space that is partially ordered and complete and $g: S \to S$ be a mapping and exists a mapping $m: S \to [0,1)$ such that the following hold:

(1) There exists $s_0 \in S$ such that $s_0 \preccurlyeq g(s_0)$;
(2) $m(g(s)) \le m(s)$ for all $s \in S$;
(3) $d_c\big(g(s), g(t)\big) \le m(s) \frac{[d_c(s, g(t)) - d_c(t, g(s))]}{2}$, for all $s, t \in S$ with $t \preccurlyeq s$;
(4) g is continuous and nondecreasing.

Then there is a fixed point of g in S. Moreover, if $g(v) = v$, then $d_c(v, v) = 0$.

In addition, the following example illustrates the main theorem.

Example 3 :

Let $X = \mathbb{R}$ dan $C = \{x \in E: x \ge 0\}$. Suppose $S = [0,1]$ and a mapping $d_c: S \times S \to X$ with $d_c(x, y) = x$ for all $x, y \in S$. Then $(S, d_c, \preccurlyeq)$ is a complete partially ordered metric *cone* space. Now, define $g: S \to S$ with $g(x) = \frac{s^2}{40}$ or all $s \in S$. Given $m(s) = \frac{s}{3}$ for all $s \in S$. We show that there is a fixed point of g in S. Moreover, if $g(v) = v$, then $d_c(v, v) = 0$.
We show that all conditions in theorem hold.

(1) Firstly, we show the existence of partially ordered elements, there exist $x_0 \in S$ such that $s_0 \preccurlyeq g(s)$. We recall, $s_0 \preccurlyeq g(s_0) = g(s_0) - s_0 \in C$ with $C = \{x \in X: x \ge 0\}$. Then

$$g(s_0) - s_0 \ge 0,$$

$$\frac{s_0^2}{40} - x_0 \ge 0,$$

$$x_0\left(\frac{x_0}{40}-1\right)\geq 0,$$
$$x_0=0 \text{ or } x_0=40.$$

Since $S=[0,1]$, choose $s_0=0$, then $g(s_0)=\frac{x_0^2}{40}=\frac{0^2}{40}=0$. So, $s_0 \preccurlyeq g(s_0)$.

(2) After that, we show $m(g(s))\leq m(s)$ for all $s\in S$. Since $g(s)=\frac{s^2}{40}$ for all $s\in S$, $m(s)=\frac{s}{3}$ for all $s\in S$, then $m\big(g(s)\big)=\frac{g(s)}{3}=\frac{\left(\frac{s^2}{40}\right)}{3}=\frac{s^2}{120}\leq\frac{s}{3}=m(s)$ for all $s\in[0,1]$.

(3) Then, we state $d_c\big(g(s),g(t)\big)\leq m(s)d_c(s,t)$ for all $s,t\in S$ with $t\preccurlyeq s$.
Given $d_c\colon S\times S\to X$ is defined by $d_c(s,t)=s$ for all $s,t\in S$. Then,
$d_c(g(s),g(t))=g(s)=\frac{s^2}{40}\leq\frac{s^2}{3}=\frac{s}{3}s=m(s)d_c(s,t)$ for all $s\in[0,1]$.

(4) Last, we show that g is continuous and non-decreasing.
Suppose g is not a non-decreasing function, then if $0\leq s<t\leq 1$ then $g(s)>g(t)$. But,

$$g(s)=\frac{s^2}{40}<\frac{t^2}{40}=g(t).$$

It is a contradiction, so g is nondecreasing. Now, we have

$$g(s^*)=s^*,$$
$$\frac{{s^*}^2}{40}=s^*,$$
$$\frac{{s^*}^2}{40}-s^*=0,$$
$$s^*\left(\frac{s^*}{40}-1\right)=0,$$
$$s^*=0 \text{ or } s^*=40.$$
$$s^*=0 \text{ is in } S, \text{ but } s^*=40 \text{ is not in } S.$$

So, $g(x)=\frac{x^2}{40}$ has a unique fixed point 0, because $g(0)=0$. Then , $d_c(s^*,s^*)=0$.

3 Conclusions

We concluded that the fixed point theorem on a complete cone metric space has four conditions that must be satisfied to obtain a unique fixed point. Then, One of the conditions in the theorem is to replace the constant into a function. As cosequence, we get three different characteristics for contraction mapping expressed in three corollaries.

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