On the sum of an idempotent and a tripotent in a quaternion algebra over the ring of integers modulo p

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Abstract. Let H be denoted as quaternions. Quaternions form an algebra over a ring R, as an extension of complex numbers into a four dimensional space, where $H = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in R\}$. A quaternion algebra, particularly defined over fields of characteristic 0, finds numerous applications in physics. In this article, we explore some properties of the sum of an idempotent and a tripotent in the finite ring H/Z_p, adapting the definition of SIT rings that was introduced by Ying et al in 2016. We provide some conditions for H/Z_p to be SIT rings and we give some examples of weakly tripotent rings (Breaz and Cimpean, 2018) in H/Z_p.

1 Introduction

Let Z_p be denoted as finite field of order p, where p is a prime. For an odd prime p, quaternion algebra over Z_p (denoted as H/Z_p) is defined to be the set $H = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in Z_p; i^2 = j^2 = k^2 = -1; ij = k, ji = -k\}$. Since the quaternion algebra is defined over Z_p, it can seen as vector space over Z_p. Hence, $H/Z_p = Z_p \oplus iZ_p \oplus jZ_p \oplus kZ_p$.In this paper '=' will be used to denote congruence '≡', that is $i^2 = -1$ actually means $i^2 \equiv -1(\text{mod}(p))$. For the case of $p = 2$ (Quaternion ring with characteristic 2), we will define $H = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in Z_2; i^2 = j^2 = k^2 = 1; ij = ji = k\}$ following [2]. This definition is not a usual definition for Quaternion algebra with characteristic 2, since quartenion algebra will be isomorphic to noncommutative division algebra over ring field F or isomorphic to 2 by 2 matrix ring over $M_2(F)$, but H/Z_p defined in this paper is commutative. For more information about Quaternion algebra with characteristic 2 that is isomorphic to $M_2(F)$, see [8]. Note that the quartenion algebra defined in this paper follows the definition of Quaternion algebra in [1], [2], and [3]. There is another way to define Quaternion algebra over Z_p which is so called split Quartenion algebra in [5].

An element x in Quaternion algebra H is said to be idempotent if x satisfies the property of $x^2 = x$. The set of all idempotents in H/Z_p will be denoted as $\text{Id}(H/Z_p)$. Note that in general, $Id(H/Z_p)$ does not form a subring in H/Z_p, it does not even form a group under componentwise addition (addition when we view H/Z_p as vector space). In the same way as idempotent, an element in Quartenion algebra H is said to be tripotent if x satisfies the property of $x^3 = x$. The set of all tripotent in H/Z_p will be denoted as $Tri(H/Z_p)$. $Tri(H/Z_p)$

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also does not form a subring or a subgroup under componentwise addition. Thus, we are interested to know the property of element x that is sum of idempotent and tripotent.

Recently study about structure of tripotent H/Z_p has been done by [2]. In this paper, we are interested in finding some property of element x which can be represent as sum of two idempotents (we called x an SII element). Further more, we are interested to find some results if we replace one of the idempotents by tripotent. In other words, we are also interested in the property of an element y which is sum of an idempotent and a tripotent (we called y an SIT element).

2 SII elements in Quartenion algebra over H/Z_p

In this section, we will assume p is a odd prime so that 2 is invertible in H/Z_p. Any structure of SII elements of H/Z_2 will be describe in Section 4. Now, we will list some of the well-known results regarding conditions for an element $x \in H/Z_p$ to be idempotent.

Proposition 1 ([2]). *Let $x \in H/Z_p$ be of the form $x = a_1 i + a_2 j + a_3 k$. Then, x is not an idempotent.*

Theorem 1 ([2]). *Let p be an odd prime and let $x = a_0 + a_1 i + a_2 j + a_3 k \in H/Z_p$. Then, x is nontrivial idempotent if $a_0 = \frac{p+1}{2}$ and $a_1^2 + a_2^2 + a_3^2 = \frac{p^2-1}{4}$.*

Clearly, 0 and 1 are trivial idempotents. Proposition 1 shows that all "purely imaginary" idempotents in H/Z_p are trivial idempotents. Hence, Theorem 1 and Proposition 1 together shows all possible idempotents in H/Z_p.

To continue with the analysis of SII elements in H/Z_p, we will first define the scalar part and imaginary part of an element $x \in H/Z_p$. After that, we will find all conditions of SII elements by checking the scalar part and imaginary part of an element in case-by-case fashion. Firstly, if we view H/Z_p as vector space, then for every element $x \in H/Z_p$, we can write x as linear combination of the standard basis $\{1, i, j, k\}$. i.e. $x = x_0 + x_1 i + x_2 j + x_3 k$. For simplicity, we will use the 4-tuple notation to denote the vector x. i.e. $x = (x_0, x_1, x_2, x_3)$. Now, we introduce the following definition.

Definition 1. *Let $x = x_0 + x_1 i + x_2 j + x_3 k$ be the standard representation of x as linear combination of standard basis in H/Z_p, we define $\mathbb{S} : H/Z_p \longrightarrow Z_p$ to be the function $\mathbb{S}(x) = x_0$ and $\mathbb{I} : H/Z_p \longrightarrow iZ_p \oplus jZ_p \oplus kZ_p$ to be the function $\mathbb{I}(x) = x_1 i + x_2 j + x_3 k$.*

We define $\mathbb{I}(x)$ as the "imaginary part" of an element $x \in H/Z_p$. In the same way we define $\mathbb{S}(x) = x_0$ as the "scalar part" of an element $x \in H/Z_p$. Clearly, $x = \mathbb{S}(x) + \mathbb{I}(x)$ and we can study the conditions for x to be a sum of two idempotents in two different components $\mathbb{S}(x), \mathbb{I}(x)$. Note that, in order to determine whether x is a nontrivial idempotent, we only need to check the scalar part condition and the imaginary part condition of x. For nontrivial idempotents in H/Z_p, the scalar condition is fixed ($x_0 = 2^{-1}$). The imaginary part condition, is the solution $\mathbb{I}(x) = (x_1, x_2, x_3)$ to the equation

$$x_1^2 + x_2^2 + x_3^2 = \frac{1}{4} \tag{1}$$

over $iZ_p \oplus jZ_p \oplus kZ_p$. We will use the notation $Q(\mathbb{I}(x)) = -1/4$ to simplify equation 1, where $Q(x) = xx^T = x_0^2 + x_1^2 + x_2^2 + x_3^2$ (we view x as a row vector and x^T is the tranpose of x). Now, we can rephrase Theorem 1 and Proposition 1 in a simpler form.

Theorem 2. *If $x \in Id(H/Z_p)$, then x is either (exclusively) one of the following forms:*

1. *(Trivial idempotent)* $\mathbb{S}(x) \in \{0, 1\}$ *and* $\mathbb{I}(x) = \underset{\sim}{0}$.
2. *(Nontrivial idempotent)* $\mathbb{S}(x) = 2^{-1}$ *and* $Q(\mathbb{I}(x)) = -2^{-2}$.

If x is sum of two nontrivial idempotents, then we say that x is a nontrivial SII element. Thus, we have Lemma 1 that state the scalar condition and imaginary condition for nontrivial SII element.

Lemma 1. *An element x in H/Z_p is nontrivial SII element ($x = a+b$, where $a, b \in Id(H/Z_p) \setminus \{0, 1\}$) if and only if x satisfies the following two conditions:*

1. *(Scalar part condition)* $\mathbb{S}(x) = 1$,
2. *(Imaginary part condition)* $Q(\mathbb{I}(a)) = Q(\mathbb{I}(b)) = -2^{-2}$.

Proof. Let $x = a+b$, where $a, b \in Id(H/Z_p)\setminus\{0, 1\}$. Then, we have $\mathbb{S}(a)+\mathbb{S}(b) = 2^{-1}+2^{-1} = 1$. For the later part, $Q(\mathbb{I}(a)) = Q(\mathbb{I}(b)) = -2^{-2}$ is just restatement of Theorem 1. □

Now we state the complete conditions for an element $x \in H/Z_p$ to be an SII elements.

Theorem 3. *An element x in H/Z_p is an SII element ($x = a + b$, where $a, b \in Id(H/Z_p)$) if and only if x satisfies one of the following conditions:*

1. $\mathbb{S}(x) \in \{0, 1, 2\}$ *and* $\mathbb{I}(x) = \underset{\sim}{0}$,
2. $\mathbb{S}(x) = 1$ *and* $\mathbb{I}(x) = \mathbb{I}(a) + \mathbb{I}(b)$ *such that* $Q(\mathbb{I}(a)) = Q(\mathbb{I}(b)) = -2^{-2}$,
3. $\mathbb{S}(x) = \{2^{-1}, 3 \cdot 2^{-1}\}$ *and* $Q(\mathbb{I}(x)) = -2^{-2}$.

In particular, we have a simple description for scalar part condition on SII element x, that is $\mathbb{S}(x) \in \{0, 1, 2^{-1}, 2, 3 \cdot 2^{-1}\}$.

Proof. We first seperate $x = a + b$, where $a, b \in Id(H/Z_p)$ into three cases.

Case 1: Let $x = a + b$, where a and b are trivial idempotents. Then, clearly we have $\mathbb{S}(x) \in \{0, 1, 2\}$. Since a, b are trivial idempotents, $\mathbb{I}(a) = \mathbb{I}(b) = 0$. Thus, $\mathbb{I}(x) = \mathbb{I}(a)+\mathbb{I}(b) = 0$.

Case 2: Let $x = a + b$, where a and b are nontrivial idempotents. Then, from Lemma 1 we have $\mathbb{S}(x) = 1$ and $Q(\mathbb{I}(a)) = Q(\mathbb{I}(b)) = -2^{-2}$.

Case 3: Let $x = a + b$, where a is trivial idempotent while b is nontrivial idempotent. Then, $\mathbb{S}(a) + \mathbb{S}(b) \in \{0, 1\} + \{2^{-1}\} = \{2^{-1}, 3 \cdot 2^{-1}\}$. The case when a is nontrivial idempotent while b is trivial idempotent is similar to this case.

Therefore, combining the three cases, we have $\mathbb{S}(x) \in \{0, 1, 2^{-1}, 2, 3 \cdot 2^{-1}\}$ as desired. □

3 SIT elements in Quartenion algebra over H/Z_p

Similar to the previous section, we first state some important results of tripotent in H/Z_p. Note that, a purely imaginary element x of H/Z_p is element with 0 in scalar part ($\mathbb{S}(x) = 0$), and the set of all purely imaginary element will be denoted as $\mathcal{P}$.

Proposition 2 ([3])**.** *Let $x \in H/Z_p$ be a pure imaginary element of the form $x = a_1 i+a_2 j+a_3 k$, where at least one of a_1, a_2, a_3 are non-zero. Then, x is tripotent if and only if* $\mathrm{a}_1^2 + \mathrm{a}_2^2 + \mathrm{a}_3^2 = \mathrm{p} - 1$.

Note that the trivial tripotents of H/Z_p are $0, 1, -1$. Hence, we defined nontrivial tripotents of H/Z_p to be tripotent that is neither trivial tripotent nor purely imaginary tripotent. The following Theorem 4 shows all the conditions for nontrivial tripotent.

Theorem 4 ([3]). *Let $x \in H/Z_p$, where p is prime and $p \neq 2, 3$, be an element of the from $x = a_0 + a_1 i + a_2 j + a_3 k$, where $a_0 \neq 0$ and at least one of a_1, a_2, a_3 is non-zero. Then, x is tripotent if and only if $a_0^2 = \frac{1-p}{4}$ and $a_1^2 + a_2^2 + a_3^2 = \frac{p-1}{4}$.*

Since $a_0^2 = \frac{1-p}{4}$ if and only if $a_0 \in \{2^{-1}, -2^{-1}\}$, we summarize the results of Proposition 2 and Theorem 4 as Theorem 5.

Theorem 5. *If $x \in Tri(H/Z_p)$, then x is either (exclusively)*

1. *(Trivial tripotent) $\mathbb{S}(x) \in \{0, 1, -1\}$ and $\mathbb{I}(x) = \underset{\sim}{0}$.*
2. *(Nontrivial tripotent) $\mathbb{S}(x) \in \{2^{-1}, -2^{-1}\}$ and $Q(\mathbb{I}(x)) = -2^{-2}$.*
3. *(Purely imaginary tripotent) $\mathbb{S}(x) = 0$ and $Q(\mathbb{I}(x)) = -1$.*

Now we look into the nontrivial SIT element in H/Z_p. That is, elements that can be written as sum of a nontrivial idempotent and a nontrivial tripotent.

Lemma 2. *An element x in H/Z_p is nontrivial SIT element ($x = a + b$, where $a \in Id(H/Z_p) \setminus \{0, 1\}$ and $b \in Tri(H/Z_p) \setminus \{0, 1, -1\} \cup Tri(\mathcal{P})$) if and only if the following two conditions hold.*

1. *(Scalar part condition) $\mathbb{S}(x) \in \{0, 1\}$,*
2. *(Imaginary part condition) $Q(\mathbb{I}(a)) = Q(\mathbb{I}(b)) = -2^{-2}$.*

Proof. Let $x = a + b$, where $a \in Id(H/Z_p) \setminus \{0, 1\}$ and $b \in Tri(H/Z_p) \setminus \{0, 1, -1\} \cup Tri(\mathcal{P})$. Then, we have $\mathbb{S}(a) + \mathbb{S}(b) \in \{2^{-1}\} + \{2^{-1}, -2^{-1}\} = \{0, 1\}$. The later part is restatement of Theorem 2 and Theorem 5. □

Later on, we look into the element in H/Z_p that is the sum of a nontrivial idempotent and a purely imaginary tripotent.

Lemma 3. *An element x in H/Z_p is SIT element such that $x = a+b$, where $a \in Id(H/Z_p) \setminus \{0, 1\}$ and $b \in Tri(\mathcal{P})$, if and only if the following two conditions hold.*

1. *(Scalar part condition)$\mathbb{S}(x) = 2^{-1}$.,*
2. *(Imaginary part condition) $Q(\mathbb{I}(a)) = -2^{-2}$ while $Q(\mathbb{I}(b)) = -1$.*

Proof. Let $x = a+b$, where $a \in Id(H/Z_p) \setminus \{0, 1\}$ and $b \in Tri(\mathcal{P})$. Then, we have $\mathbb{S}(a)+\mathbb{S}(b) = 2^{-1} + 0 = 2^{-1}$. The imaginary part condition is just restatement of Theorem 2 and Theorem 5. □

Theorem 6. *An element x in H/Z_p is SIT element ($x = a + b$, where $a \in Id(H/Z_p)$ and $b \in Tri(H/Z_p)$), if and only if x satisfies one of the following conditions:*

1. *$\mathbb{S}(x) \in \{-1, 0, 1, 2\}$ and $\mathbb{I}(x) = \underset{\sim}{0}$,*
2. *$\mathbb{S}(x) \in \{0, 1\}$ and $\mathbb{I}(x) = \mathbb{I}(a) + \mathbb{I}(b)$ such that $Q(\mathbb{I}(a)) = Q(\mathbb{I}(b)) = -2^{-2}$,*
3. *$\mathbb{S}(x) \in \{0, 1\}$ and $Q(\mathbb{I}(x)) = -1$,*
4. *$\mathbb{S}(x) \in \{-2^{-1}, 2^{-1}, 3 \cdot 2^{-1}\}$ and $Q(\mathbb{I}(x)) = -2^{-2}$*
5. *$\mathbb{S}(x) = 2^{-1}$ and $\mathbb{I}(x) = \mathbb{I}(a) + \mathbb{I}(b)$ such that $Q(\mathbb{I}(a)) = -1$ while $Q(\mathbb{I}(b)) = -2^{-2}$*

In particular, we have a simple description for scalar part condition on SIT element x, that is $\mathbb{S}(x) \in \{-1, 0, 1, 2, 2^{-1}, -2^{-1}, 3 \cdot 2^{-1}\}$.

Proof. We first seperate $x = a + b$, where $a \in Id(H/Z_p)$ and $b \in Tri(H/Z_p)$ into six cases.

Case 1: Let $x = a+b$, where a is trivial idempotent and b is trivial tripotent. Then, clearly we have $\mathbb{S}(x) \in \{-1, 0, 1, 2\}$ and $\mathbb{I}(x) = \underset{\sim}{0}$.

Case 2: Let $x = a+b$, where a is nontrivial idempotent and b is nontrivial tripotent. Then, by Lemma 2 we have $\mathbb{S}(x) \in \{0, 1\}$ and $Q(\mathbb{I}(a)) = Q(\mathbb{I}(b)) = -2^{-2}$.

Case 3: Let $x = a + b$, where a is trivial idempotent while b is purely imaginary. Then, $\mathbb{S}(a) + \mathbb{S}(b) \in \{0, 1\} + \{0\} = \{0, 1\}$ and $Q(\mathbb{I}(x)) = -1$.

Case 4: Let $x = a+b$, where a is trivial idempotent while b is nontrivial idempotent. Then, $\mathbb{S}(a) + \mathbb{S}(b) \in \{0, 1\} + \{2^{-1}, -2^{-1}\} = \{-2^{-1}, 2^{-1}, 3 \cdot 2^{-1}\}$. For the imaginary part, since we have $Q(\mathbb{I}(b)) = -2^{-2}$ and $\mathbb{I}(a) = \underset{\sim}{0}$, thus $Q(\mathbb{I}(x)) = Q(\mathbb{I}(a) + \mathbb{I}(b)) = Q(\underset{\sim}{0} + \mathbb{I}(b)) = Q(\mathbb{I}(b)) = -2^{-2}$.

Case 5: Let $x = a + b$, where a is nontrivial idempotent while b is trivial tripotent. Then, $\mathbb{S}(a) + \mathbb{S}(b) \in \{2^{-1}\} + \{0, 1, -1\} = \{-2^{-1}, 2^{-1}, 3 \cdot 2^{-1}\}$. Since we have $Q(\mathbb{I}(a)) = -2^{-2}$ and $\mathbb{I}(b) = \underset{\sim}{0}$, thus $Q(\mathbb{I}(x)) = Q(\mathbb{I}(a) + \mathbb{I}(b)) = Q(\mathbb{I}(a) + \underset{\sim}{0}) = Q(\mathbb{I}(b)) = -2^{-2}$. This case is similar to Case 4.

Case 6: Let $x = a+b$, where a is nontrivial idempotent while b is purely imaginary. Then, by Lemma 3 we have $\mathbb{S}(x) = 2^{-1}$ and $Q(\mathbb{I}(a)) = -1$ while $Q(\mathbb{I}(b)) = -2^{-2}$.

Therefore, combining the six cases, we have $\mathbb{S}(x) \in \{-1, 0, 1, 2, 2^{-1}, -2^{-1}, 3 \cdot 2^{-1}\}$ as desired. □

4 SIT Quartenion algebra over H/Z_p

In this paper, a quaternion algebra will be called an Strong SIT quaternion algebra if all elements of the quaternion algebra are commuting SIT elements following the definition of Strong SIT rings in [7]. That is, if $x = e + t$ where $e \in Id(H/Z_p)$ and $t \in Tri(H/Z_p)$, then the idempotent and the tripotent will commute ($et = te$). Thus, we have the following theorem about Quartenion algebra over Z_p that is also a Strong SIT ring.

Theorem 7. *H/Z_2 is the only Strong SIT quartenion algebra over Z_p.*

Proof. First, we need to show that H/Z_2 is indeed an SIT ring. H/Z_2 defined in section 1 is a commutative ring (every idempotent will commute with tripotent), hence we only need to show that every element in H/Z_2 can be written as sum of an idempotent and a tripotent. Note that, every element $x = x_0 + x_1 i + x_2 j + x_3 k$ in H/Z_2 has the following property:

$$(x_0 + x_1 i + x_2 j + x_3 k)^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0 + x_1 + x_2 + x_3. \tag{2}$$

It is clear from equation 2 that an element x in H/Z_2 is nilpotent if x has even number of nonzero components, while x is an order 2 unit (thus a tripotent) if x has odd number of nonzero components.

Now, we consider the case when x has odd number of nonzero components. It is clear that x is sum of an idempotent and a tripotent by considering 0 as the idempotent and x as tripotent (every order 2 unit is a tripotent).

We proceed with the remaining case. If x has even number of nonzero components, then x either has 2 nonzero components or $x = 1+i+j+k$. For the both of the cases, $x = 1+(1+x)$ is sum of 1 (an idempotent) and $1 + x$ (a tripotent because $1 + x$ has odd number of nonzero components).

The nonexistence of SIT quartenion algebra over Z_p when p is odd prime clear from the facts that H/Z_p is isomorphic to $M_2(Z_p)$ and Example 3.1 in [7]. □

Follow from the above question, it is natural to ask whether there is SII or SIT quaternion algebra (dropping the Strong property). Therefore, we have the following Corollary 1 and Corollary 2 which are direct consequence of Theorem 3 and Theorem 6 respectively.

Table 1. Partition of H/Z_3 according to trivial idempotents, trivial tripotents, nontrivial idempotents, nontrivial tripotents, purely imaginary tripotents, sum of idempotent and tripotent, and sum of two tripotents

Type	Elements of the partition
Trivial idempotents	(0,0,0,0),(1,0,0,0)
Trivial tripotents	(2,0,0,0)
Nontrivial idempotents	(2,0,1,1),(2,0,1,2),(2,0,2,1),(2,0,2,2),(2,1,0,1),(2,1,0,2), (2,1,1,0),(2,1,2,0),(2,2,0,1),(2,2,0,2),(2,2,1,0),(2,2,2,0)
Nontrivial tripotents	(1,0,1,1),(1,0,1,2),(1,0,2,1),(1,0,2,2),(1,1,0,1),(1,1,0,2), (1,1,1,0),(1,1,2,0),(1,2,0,1),(1,2,0,2),(1,2,1,0),(1,2,2,0)
Purely imaginary tripotents	(0,0,1,1),(0,0,1,2),(0,0,2,1),(0,0,2,2),(0,1,0,1),(0,1,0,2), (0,1,1,0),(0,1,2,0),(0,2,0,1),(0,2,0,2),(0,2,1,0),(0,2,2,0)
Sum of idempotent and idempotent	(1,0,0,1),(1,0,0,2),(1,0,1,0),(1,0,2,0),(1,1,0,0),(1,1,1,1), (1,1,1,2),(1,1,2,1),(1,1,2,2),(1,2,0,0),(1,2,1,1),(1,2,1,2), (1,2, 2,1),(1,2,2,2)
Sum of idempotent and tripotent	(0,0,0,1),(0,0,0,2),(0,0,1,0),(0,0,2,0),(0,1,0,0),(0,1,1,1), (0,1,1,2),(0,1,2,1),(0,1,2,2),(0,2,0,0),(0,2,1,1),(0,2,1,2), (0,2,2,1),(0,2,2,2),(2,0,0,1),(2,0,0,2),(2,0,1,0),(2,0,2,0), (2,1,0,0),(2,1,1,1),(2,1,1,2),(2,1,2,1),(2,1,2,2),(2,2,0,0), (2,2,1,1),(2,2,1,2),(2,2,2,1),(2,2,2,2)

Corollary 1. *Let H/Z_p be an quaternion algebra over Z_p. Then, H/Z_p is not an SII quaternion algebra for odd prime $p > 5$.*

Corollary 2. *Let H/Z_p be an quaternion algebra over Z_p. Then, H/Z_p is not an SIT quaternion algebra for odd prime $p > 7$.*

By Corollary 1 and Corollary 2, we only left with $p = 3, 5, 7$. Table 1 will show that every element in H/Z_3 is either a idempotent or tripotent or sum of two idempotents or sum of idempotent and tripotent. Thus, this show that H/Z_3 is a SIT ring.

For $p = 5$, by using Theorem 3 and Theorem 6 it can be shown that $2 + 2k$ is neither idempotent, tripotent, sum of two idempotents, nor sum of an idempotent and a tripotents. Similary, for $p = 7$, the element $3+k$ is neither idempotent, tripotent, sum of two idempotents, nor sum of an idempotent and a tripotent. Therefore, the only SIT quaternion algebra are H/Z_2 and H/Z_3. Furthermore, from table 1 and the fact that H/Z_2 is not a SII ring, we can infer that there is no SII quaternion algebra.

5 Conclusion

Theorem 3 and Theorem 6 show the conditions for an element x to be sum of two idempotent, and sum of an idempotent and a tripotent respectively. The lists of conditions provided are indeed complete since Theorem 2 and Theorem 5 included every possible cases of idempotents and tripotents in H/Z_p. However, some of the conditions in Theorem 3 and Theorem 6 not only involve x, but we need to know the decomposition of x as sum of an idempotent and a tripotent ($x = a + b$, $a \in Id(H/Z_p)$ and $b \in Tri(H/Z_p)$) in order to determine whether x is SII element or SIT element. Which is a circular argument if we already know x is sum of an idempotent and tripotent beforehand. However, this problem can be solved if we can strengthen Lemma 1, Lemma 2, and Lemma 3 by lifting all of the imaginary part conditions. We conjecture that Lemma 1, Lemma 2, and Lemma 3 can be simplify to only depend on the scalar part condition (wihtout any conditions in the imaginary part). Which is equivalent to the following formulation.

Conjecture. *Let $\mathfrak{Q}_a$ denote the set of solutions to the equation $Q(\mathbb{I}(x)) = a$, where $\mathbb{I}(x) \in iZ_p \oplus jZ_p \oplus kZ_p$ and $a \in Z_p$. Let $\mathfrak{Q}_a + \mathfrak{Q}_b = \{v + w | v \in \mathfrak{Q}_a; w \in \mathfrak{Q}_b\}$ denote the addition of two sets. Then,*

$$\mathfrak{Q}_{-1} + \mathfrak{Q}_{-2^{-2}} = \mathfrak{Q}_{-2^{-2}} + \mathfrak{Q}_{-2^{-2}} = iZ_p \oplus jZ_p \oplus kZ_p.$$

If this conjecture is true then we will have a complete description of SII element and SIT element in quaternion algebra as follows.

Theorem 8. *(Complete conditions for SII elements in quaternion algebra) An element x in H/Z_p is an SII element ($x = a + b$, where $a, b \in Id(H/Z_p)$) if and only if x satisfies one of the following conditions:*

1. $\mathbb{S}(x) \in \{0, 2\}$ *and* $\mathbb{I}(x) = \underset{\sim}{0}$,
2. $\mathbb{S}(x) = 1$,
3. $\mathbb{S}(x) = \{2^{-1}, 3 \cdot 2^{-1}\}$ *and* $Q(\mathbb{I}(x)) = -2^{-2}$.

In particular, we have a simple description for scalar part condition on SII element x, that is $\mathbb{S}(x) \in \{0, 1, 2^{-1}, 2, 3 \cdot 2^{-1}\}$.

Theorem 9. *(Complete conditions for SIT elements in quaternion algebra) An element x in H/Z_p is SIT element ($x = a + b$, where $a \in Id(H/Z_p)$ and $b \in Tri(H/Z_p)$), if and only if x satisfies one of the following conditions:*

1. $\mathbb{S}(x) \in \{-1, 2\}$ *and* $\mathbb{I}(x) = \underset{\sim}{0}$,
2. $\mathbb{S}(x) \in \{0, 1\}$,
3. $\mathbb{S}(x) \in \{-2^{-1}, 3 \cdot 2^{-1}\}$ *and* $Q(\mathbb{I}(x)) = -2^{-2}$,
4. $\mathbb{S}(x) = 2^{-1}$.

In particular, we have a simple description for scalar part condition on SIT element x, that is $\mathbb{S}(x) \in \{-1, 0, 1, 2, 2^{-1}, -2^{-1}, 3 \cdot 2^{-1}\}$.

Acknowledgement

The authors would like to express their sincere gratitude to the Center for Mathematical Sciences, Universiti Tunku Abdul Rahman, for their generous financial support (vote number 4342/002). Additionally, we extend our appreciation to Dagang Network Holding sdn bhd for their valuable grant (vote number 8167/0001), which has significantly contributed to the research presented in this paper. Their support has been instrumental in enabling the completion of this study.

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