Smallest Cubic Graphs with Given Girth and Skewness

Chii Liang Ng[1,*], *Gek L.* Chia[1,**], and *Denis Chee-Keong* Wong[1]

[1]Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, Sungai Long Campus, Cheras 43000 Kajang, Selangor, Malaysia

Abstract. The skewness of a graph is the minimum number of its edges whose deletion results in a planar graph. We determine the minimum orders of cubic graphs with girth no more than 6 and skewness no more than 4. In passing, we also determine the skewness of all cubic cages whose girth is no more than 8.

1 Introduction

All graphs considered in this paper are finite, simple, and connected unless otherwise stated. For any graph G, the *skewness* of G, denoted as $sk(G)$, is the minimum number of edges in G whose deletion yields a planar graph. A set S of edges in G is called a *skewness set* of G if $|S| = sk(G)$, and deleting the edges in S from G yields a planar graph.

In the early 1970s, the concept of skewness was considered independently by Guy [8] and Kainen [9]. Kainen [9] also proved the following lower bound for the skewness of connected graphs.

Theorem 1.1 ([9])**.** *Suppose G is a connected graph with p vertices, q edges, and of girth g. Then,* $sk(G) \geq \lceil q - \frac{g}{g-2}(p-2) \rceil$.

The problem of finding the skewness of a graph is known to be NP-complete [10], the cubic variant of this problem is also NP-complete [6].

Another important measure for non-planarity of a graph is the notion of *crossing number*. The crossing number of a graph G, denoted as $cr(G)$, is the minimum number of edge intersections in a plane drawing of G. Clearly, $cr(G) \geq sk(G)$ for any graph G. The problem of deciding the crossing number of a graph is also NP-complete [7].

One research focus of the crossing numbers of graphs involves searching for the smallest regular graphs with given girths and crossing numbers. For small crossing numbers, the search was done manually [1, 11]. However, as the crossing number increases, the task becomes increasingly intractable due to its NP-completeness. With the assistance of the computer, the searching was done up to crossing numbers 11 for cubic graphs [4, 12].

On the other hand, there is much less research on searching for the smallest regular graphs with specific skewness. Only some upper bounds on the minimum number of vertices for large skewness were proven in [13].

By an (r, g)-*graph* we mean an r-regular graph with girth g. An (r, g)-graph with the minimum number of vetices is known as an (r, g)-*cage*. If an (r, g)-graph has skewness s, we call it an $(r, g : s)$-*graph*.

[*]e-mail: ng1902147@1utar.my
[**]e-mail: chiagl@utar.edu.my

The primary objective of this paper is to determine the minimum number of vertices in $(3, g : s)$-graphs where $s \leq 4$.

For this purpose, we use $v(r, g : s)$ to denote the minimum number of vertices for a $(3, g : s)$-graph. The case where $s = 0$ is identical to the case with crossing number 0, and is available in [1]. Hence, we shall assume that $s \geq 1$. In Section 2, we determine $v(3, g : s)$ for $s = 1, 2, 3$. Subsequently, in Section 3, we determine the order of the smallest cubic graphs with skewness 4.

In Section 4, we determine the skewness of $(3, g)$-cages for all values of $g \leq 8$. It is interesting to note that the skewness of all these cages attain the lower bound of Theorem 1.1. Graphs which attain the lower bound of Theorem 1.1 are termed π-skew (see [3]).

The following remark shall be used very often in the subsequent sections.

Remark 1.

(i) If a graph G_1 is a subdivision of another graph G, then $sk(G_1) = sk(G)$.

(ii) Suppose G is a cubic graph with p vertices. Then, the number of edges in G is $3p/2$. Also, let G_2 be a graph obtained from G by replacing a vertex with a triangle, then $sk(G_2) = sk(G)$.

(iii) If G is a cubic graph with girth 6, then it follows from Theorem 1.1 that $sk(G) \geq 3$.

If G is a connected plane graph, we let f_k denote the number of faces bounded by precisely k edges of G.

The proof of the following lemma is omitted since it follows easily from the fact that each edge is counted twice when counting the number of edges bounding a face.

Lemma 1.2. *Suppose G is a connected plane graph with girth g. Then,* $2|E(G)| = \sum_{k \geq g} k f_k$.

2 Smallest Cubic Graphs of Skewness 1, 2, and 3

In this section, we determine $v(3, g : s)$ for $s = 1, 2, 3$.

Proposition 2.1. $v(3, 3 : 1) = 8$, $v(3, 4 : 1) = 6$, $v(3, 5 : 1) = 14$, $v(3, 3 : 2) = 12$, $v(3, 4 : 2) = 12$, *and* $v(3, 5 : 2) = 10$. $(3, g : 1)$-*graph and* $(3, g : 2)$-*graph do not exist for* $g \geq 6$.

Proof. The only non-planar cubic graph of order no more than 6 is the utility graph $K_{3,3}$, with skewness 1. Hence $v(3, 4 : 1) = 6$.

Note that we can obtain a $(3, 3 : 1)$-graph by replacing a vertex in $K_{3,3}$ with a triangle (see Figure 1(a)). Hence, $v(3, 3 : 1) = 8$.

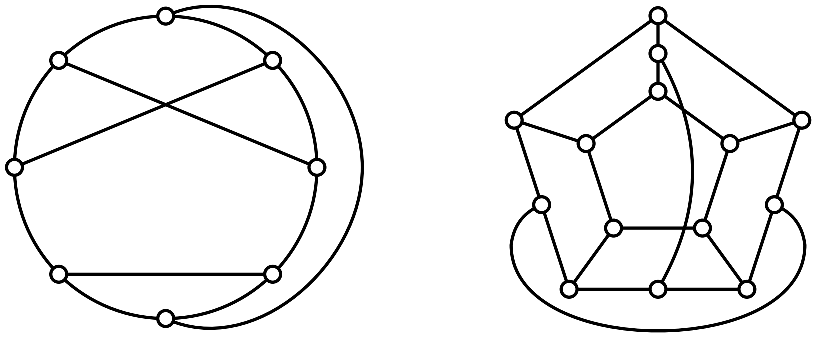
Consider the case where the cubic graph has girth 5 and skewness 1. It follows from Theorem 1.1 that the number of vertices is at least 14. Hence $v(3, 5 : 1) \geq 14$. Moreover, the lower bound is achieved in this case as shown by the cubic graph of order 14 in Figure 1(b).

In [1, Section 5.1], the authors remarked that the Petersen graph is the only graph of order no more than 10 with skewness 2. Furthermore, some $(3, 3 : 2)$-graphs and $(3, 4 : 2)$-graphs of order 12 were also listed. Some of these examples are depicted in Figure 2. Hence, we have $v(3, 3 : 2) = 12$, $v(3, 4 : 2) = 12$, and $v(3, 5 : 2) = 10$.

By Remark 1(iii), $(3, g : 1)$-graph and $(3, g : 2)$-graph do not exist for $g \geq 6$.

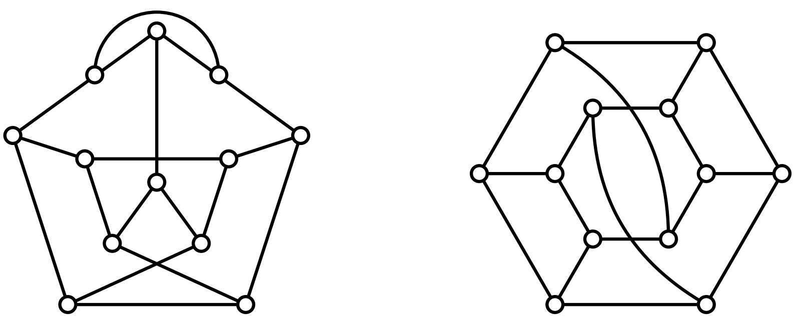
This completes the proof. □

Proposition 2.2. $v(3, 3 : 3) = 16$, $v(3, 4 : 3) = 16$, $v(3, 5 : 3) = 16$, *and* $v(3, 6 : 3) = 14$. *No* $(3, g : 3)$-*graph exists for* $g \geq 7$.



(a) $g = 3$ (b) $g = 5$

Figure 1: Smallest $(3, g : 1)$-graphs.



(a) $g = 3$ (b) $g = 4$

Figure 2: Smallest $(3, g : 2)$-graphs.

Proof. Suppose G is a cubic graph.

Since $sk(G) = 3$ in this case, we just need to consider graphs with crossing number at least 3.

Pegg Jr. and Exoo [12] verified that the minimum number of vertices in a cubic graph having crossing number 3 is 14, and there are only 8 such cubic graphs. It has been verified in [13] that except for the Heawood graph, all have skewness 2.

Since the Heawood graph is of girth 6, by Remark 1(iii), its skewness is at least 3. The drawing in Figure 3(a) shows that it has skewness at most 3. Hence, we have $\upsilon(3, 6 : 3) = 14$.

By Remark 1(ii), the cubic of order obtained from the Heawood graph by replacing a vertex with a triangle has skewness 3. Hence, we have $\upsilon(3, 3 : 3) = 16$.

Let H_1 be the graph obtained from the Heawood graph (Figure 3(a)) by subdividing the edges x_1x_2, x_3x_4 with 2 new vertices and join them with a new edge. Clearly, H_1 (Figure 3(b)) is of girth 4 and having skewness 3. Hence, we have $\upsilon(3, 4 : 3) = 16$.

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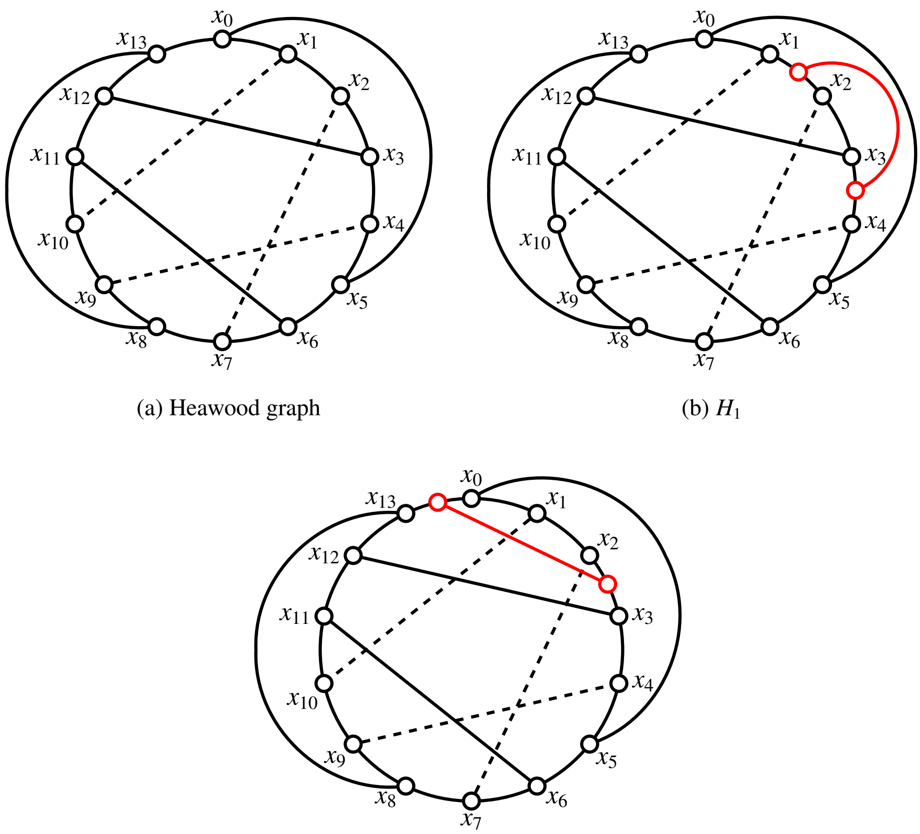

(a) Heawood graph (b) H_1

(c) H_2

Figure 3: Smallest $(3, g : 3)$-graphs with skewness set $\{x_1x_{10}, x_2x_7, x_4x_9\}$.

Let H_2 be the graph obtained from the Heawood graph (Figure 3(a)) by subdividing the edges x_0x_{13}, x_2x_3 with 2 new vertices and join them with a new edge. Clearly, H_2 (Figure 3(c)) is of girth 5 and having skewness 3. Hence, we have $v(3, 5 : 3) = 16$.

Now consider the case where G has girth 7. By Theorem 1.1 (and using Remark 1(ii)), we have $sk(G) \geq 5$. This proves the last statement of the proposition, hence the proof is complete.

□

3 Smallest Cubic Graph of Skewness 4

This section is devoted to determining the order of the smallest cubic graphs with skewness 4.

Consider the cubic graph in Figure 4(a) which is known as the *Möbius-Kantor* graph. Since it has girth 6, by Remark 1(iii), it has skewness at least 3. The drawing in Figure 4(b) shows that $S = \{x_0x_{15}, x_1x_{12}, x_3x_{14}\}$ is a skewness set of the Möbius-Kantor graph. This proves that the Möbius-Kantor graph has skewness 3.

Let $MK+$ denote the graph obtained from the Möbius-Kantor graph by subdividing the edges $x_6x_7, x_{14}x_{15}$ each with a new vertex (z and y resp.) and joining y and z with a new edge yz.

Clearly, $sk(MK+) \leq 4$ as we can obtain a spanning planar subgraph by deleting the edges in $S \cup \{yz\}$.

We first prove that $MK+$ has skewness 4 and then proceed to show that it is in fact one of the smallest cubic graphs.

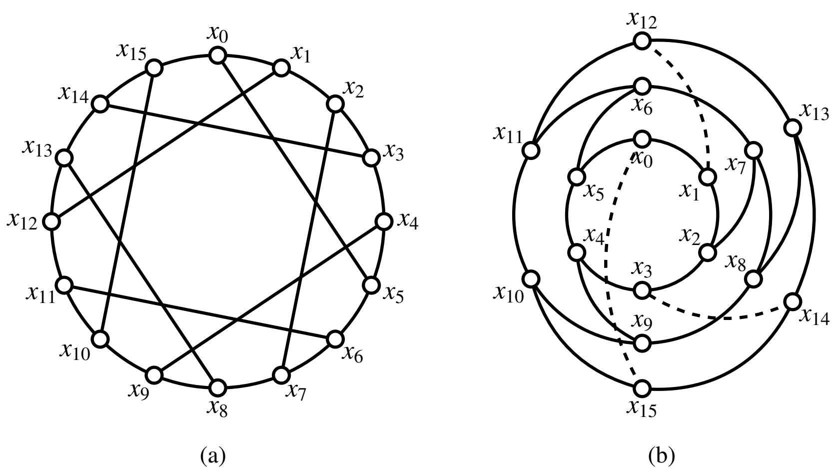

Figure 4: Two drawings of the Möbius-Kantor graph.

Lemma 3.1. $sk(MK+) = 4$.

Proof. We prove this by contradiction.

Suppose on the contrary that $sk(MK+) = 3$. Let S be a skewness set of $MK+$ with $|S| = 3$. Also, let X be the spanning planar subgraph obtained from $MK+$ by deleting all edges in S. Then, X is a hexagulation. More precisely, X can be redrawn as a plane graph with 18 vertices, 24 edges, and 8 faces. Since the number of faces is $f_6 + f_7 + \ldots$, it follows from Lemma 1.2 that each face is bounded by exactly 6 edges.

Note that the edge yz is not in S since $MK+ - yz$ is a subdivision of the Möbius-Kantor graph, which has skewness 3. Hence, yz is an edge of X.

Upon careful observation, one may notice that there are only four 6-cycles in $MK+$ containing the edge yz, namely $Z_1 = yx_{15}x_0x_5x_6zy$, $Z_2 = yx_{14}x_{13}x_8x_7zy$, $Z_3 = yx_{15}x_{10}x_{11}x_6zy$, and $Z_4 = yx_{14}x_3x_2x_7zy$.

Since every edge in X is contained in precisely two 6-faces, two of the four 6-cycles containing yz are in X.

Moreover, observe that if Z_1 is in X, then so is Z_3, since these are the only two 6-cycles that contain the edges yx_{15} and zx_6.

Likewise, if Z_2 is in X, then so is Z_4, since these are the only two 6-cycles that contain the edges yx_{14} and zx_7.

By symmetry, we can assume without loss of generality that Z_2 and Z_4 are in X.

But this implies that $\{yx_{15}, zx_6\} \subset S$, since they are not contained in any face in X.

Let Y be a subgraph of $MK+$ obtained by deleting yx_{15}, zx_6, and x_2x_3. From Figure 5, we can see that Y is a subdivision of the graph in Figure 2(b), which has skewness 2. This implies that $sk(MK+) \geq |\{yx_{15}, zx_6\}| + sk(Y) = 4$, which contradicts the assumption that $sk(MK+) = 3$. □

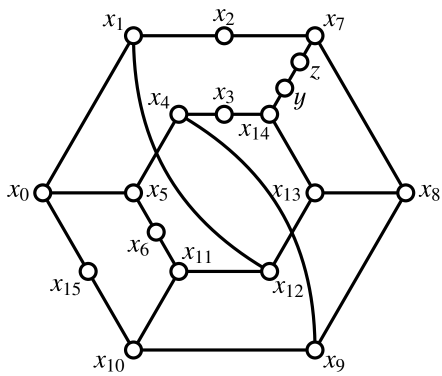

Figure 5: The subgraph Y.

Theorem 3.2. *The smallest cubic graph with skewness* 4 *is of order* 18.

Proof. Suppose G is a smallest cubic graph with $sk(G) = 4$. Then $cr(G) \geq 4$.

Pegg Jr. and Exoo [12] verified that the minimum number of vertices in a cubic graphs having crossing number 4 is 16. There are only two such graphs, i.e., the Möbius-Kantor graph and the 8-crossed prism graph, both have skewness at most 3 [13]. Hence, a cubic graph with skewness 4 cannot have fewer than 18 vertices.

By Lemma 3.1, $MK+$ is a cubic graph of order 18 having skewness 4.

This completes the proof. □

4 Skewness of $(3, g)$-cages

This section is devoted to determining the skewness of $(3, g)$-cages for $g = 3, 4, \ldots, 8$. The unique $(3, g)$-cages for $g = 3, 4, \ldots, 8$ are K_4, $K_{3,3}$, the Petersen graph, the Heawood graph, the McGee graph, and the Tutte-Coxeter graph, respectively [5].

Proposition 4.1. $(3, g)$*-cages are* π*-skew for* $g = 3, 4, \ldots, 8$.

Proof. It is easy to see that K_4 and $K_{3,3}$ are π-skew. The Petersen graph is also π-skew since its skewness is 2 (see [2]). The Heawood graph is π-skew by Proposition 2.2.

By Theorem 1.1, the lower bounds of the skewness for the McGee graph and the Tutte-Coxeter graph are 6 and 8, respectively. The drawings depicted in Figure 6 and 7(c) respectively show that the skewness of the McGee graph and the Tutte-Coxeter graph attain the lower bound.

This completes the proof. □

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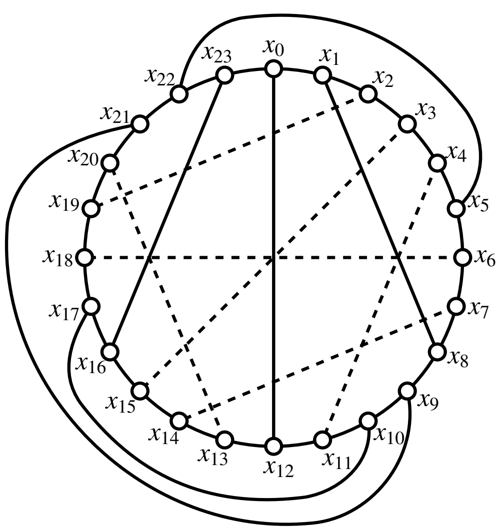

Figure 6: The McGee graph.

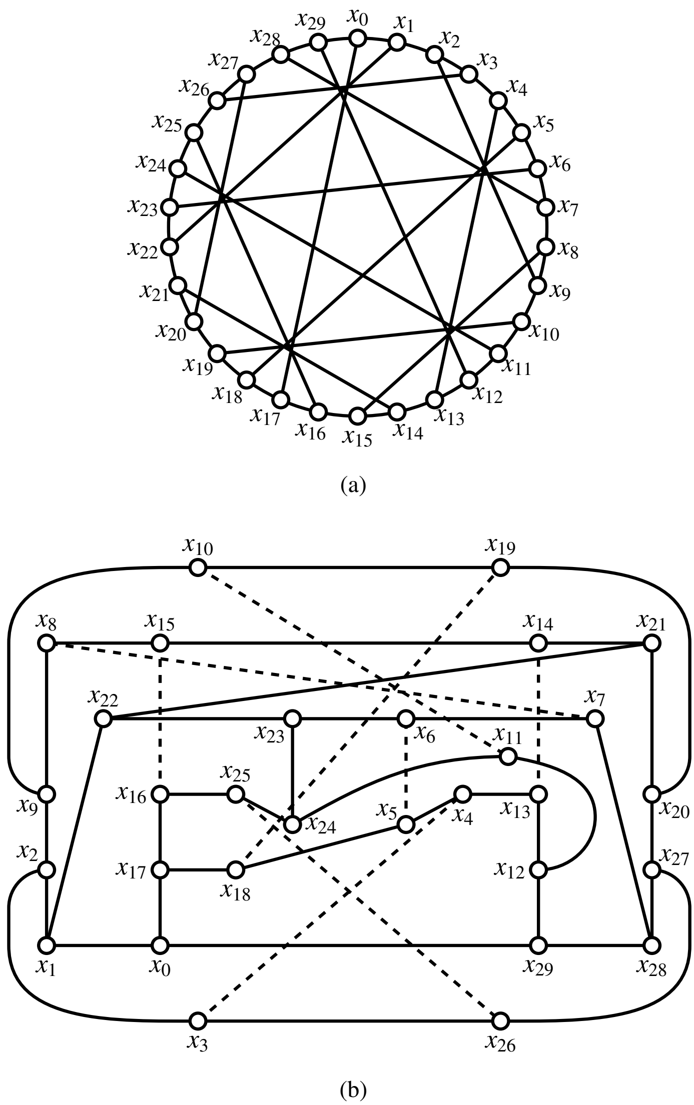

Figure 7: Two drawings of the Tutte-Coxeter graph.