Weakly Tripotent Elements in Quaternion Rings over $\mathbb{Z}_p$

Wei Jun Lee[1*], *Kiat Tat Qua*[1], and Hong Keat Yap[1]

[1]Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, 43000 Kajang, Selangor, Malaysia.

Abstract. Let R be a ring. An element $x \in R$ is weakly tripotent if x can be written as $x^3 = x$ or $x^3 = -x$. In this paper, we discuss weakly tripotency in quaternion rings over $\mathbb{Z}_p$, $\mathbb{H}/\mathbb{Z}_p$. We also give some conditions for the element $x \in \mathbb{H}/\mathbb{Z}_p$ to be weakly tripotent.

1 Introduction

In the paper of Danchev [1], an element x of ring R is said to be weakly tripotents if $x^3 = x$ or $x^3 = -x$. A ring R is called weakly tripotent if all of its elements are weakly tripotent. In [2], Aristidou and Demetre provide examples and they also establish some conditions for idempotency in $\mathbb{H}/\mathbb{Z}_p$, where p is prime. Besides that, Aristidou and Hailemariam (in [3]) used the similar methods in [2], they able to give some conditions for tripotency of quaternion rings over $\mathbb{Z}_p$ (p is prime). In this paper, we give some conditions for a quaternion ring over $\mathbb{Z}_p$ (p is prime) to be weakly tripotent.

2 Weakly tripotent elements $\mathbb{H}/\mathbb{Z}_p$

A set of real quaternions, we denote it as $\mathbb{H}$, was first introduced by Hamilton [4] in 1866 as an extension of complex number into four dimensions. A quaternion can be written as the form $x = x_0 + x_1 i + x_2 j + x_3 k$ where x_n are reals and i, j, k are complex elements such that $i^2 = j^2 = k^2 = ijk = -1$. Note that $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$. When multiplying a pair of quaternions $x = x_0 + x_1 i + x_2 j + x_3 k$ and $y = y_0 + y_1 i + y_2 j + y_3 k$, we can compute the product xy as

$$\begin{aligned} xy = &(x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) \\ &+(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)i \\ &+(x_0y_2 + x_2y_0 + x_1y_3 - x_3y_1)j \\ &+(x_0y_3 + x_3y_0 + x_1y_2 - x_2y_1)k. \end{aligned}$$

For the case, $x \in \mathbb{H}$ and $x = a_0 + a_1 i + a_2 j + a_3 k$, where $a_0 \neq 0$, $a_1 = a_2 = a_3 = 0$. The only weakly tripotent elements are $x = -1, 0$ and 1. We are focus on whether some weakly

*Corresponding author: yaphk@utar.edu.my

tripotent elements are in the quaternion rings over $\mathbb{Z}_p$. We first give the following two propositions and prove that the element x is weakly tripotent if the conditions stated in the proposition have been satisfied.

Proposition 1. Let $x \in \mathbb{H}/\mathbb{Z}_p$ be a quaternion element of the form $x = a_0 + a_1 i$, where a_0 and a_1 are nonzeros. Then $x^3 = x$ if and only if ${a_0}^2 = \frac{1-p}{4}$ and ${a_1}^2 = \frac{p-1}{4}$ and $x^3 = -x$ if and only if ${a_0}^2 = \frac{p-1}{4}$ and ${a_1}^2 = \frac{1-p}{4}$, where p is prime and $p \neq 2,3$.

Proof. Let $x = a_0 + a_1 i$ and x is weakly tripotent in $\mathbb{H}/\mathbb{Z}_p$. Then $x^3 = x$ or $x^3 = -x$.
For $x^3 = x$, by Proposition 1 in [3], it follows that $a_0^2 = \frac{1-p}{4}$ and ${a_1}^2 = \frac{p-1}{4}$.
For $x^3 = x$, we have, $(a_0 + a_1 i)^3 = -(a_0 + a_1 i)$. This follows that ${a_0}^3 - 3a_0{a_1}^2 + (3{a_0}^2 a_1 - {a_1}^3)i = -a_0 - a_1 i$. Thus, we have these two equations:

$$
\begin{gathered}
{a_0}^3 - 3a_0{a_1}^2 = -a_0 \\
3{a_0}^2 a_1 - {a_1}^3 = -a_1 .
\end{gathered}
$$

Simplify the above equations, we will obtain

$$
\begin{aligned}
{a_0}^2 - 3{a_1}^2 &= -1 \quad (1) \\
{a_0}^2 - {a_1}^2 &= -1 \quad (2)
\end{aligned}
$$

By solving the equations (1) and (2), we have

$$
{a_0}^2 - 3{a_1}^2 = 3{a_0}^2 - {a_1}^2 \Rightarrow -2{a_0}^2 = 2{a_1}^2 \Rightarrow {a_0}^2 = -{a_1}^2 \text{ ------(3).}
$$

By substitute ${a_0}^2 = -{a_1}^2$ into (1) and we get

$$
-{a_1}^2 - 3{a_1}^2 = -1 \Rightarrow -4{a_1}^2 = -1 \Rightarrow {a_1}^2 = \frac{1}{4} \text{ ------(4).}
$$

Since $p = 0 \pmod p$, ${a_1}^2 = \frac{1-p}{4}$. Substitute ${a_1}^2$ into (3) and we get ${a_0}^2 = 4^{-1}(p-1)$.
To see if the quantities $\frac{1-p}{4}$ and $\frac{p-1}{4}$ are squares mod p, we calculate the Legendre Symbol for $\frac{\frac{1-p}{4}}{p}$ and $\frac{\frac{p-1}{4}}{p}$ respectively. The first gives:

$$
\left(\frac{\frac{1-p}{4}}{p}\right) = 1
$$

and the second gives:

$$
\left(\frac{\frac{p-1}{4}}{p}\right) = \left(\frac{p-1}{p}\right)\left(\frac{\frac{1}{4}}{p}\right) = \left(\frac{p-1}{p}\right)\cdot 1 = (p-1)^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}}
$$
$$
= \begin{cases} 1, & \text{if } p \equiv 1 (\text{mod } 4) \\ -1, & \text{if } p \equiv 3 (\text{mod } 4) \end{cases}.
$$

Hence, there are no weakly tripotents of the form $a_0 + a_1 i$, if $p \equiv 3 \pmod 4$. It follows that if weakly tripotent element exists in the form of $a_0 + a_1 i$ with a_0 and a_1 both nonzero, then $p \equiv 1 (\text{mod } 4)$ and in this case, ${a_0}^2 = \frac{p-1}{4}$ and ${a_1}^2 = \frac{1-p}{4}$. Therefore, existence of weakly tripotent elements may classify the nature of p. For the converse, given ${a_0}^2 = \frac{p-1}{4}$and ${a_1}^2 = \frac{1-p}{4}$, we have that:

$$\begin{aligned} x^3 = (a_0 + a_1 i)^3 &= {a_0}^3 - 3a_0{a_1}^2 + (3{a_0}^2 a_1 - {a_1}^3)i \\ &= a_0({a_0}^2 - 3{a_1}^2) + a_1(3{a_0}^2 - {a_1}^2)i \\ &= a_0\left(\frac{p-1}{4} - 3\frac{1-p}{4}\right) + a_1\left(3\frac{p-1}{4} - \frac{1-p}{4}\right)i \\ &= a_0(p-1) + a_1(p-1)i \\ &= -a_0 - a_1 i, \text{as } p = 0 \ (\text{mod } p) \\ &= -x\,. \end{aligned}$$

Hence, x is weakly tripotent. Similarly, in Proposition 1 (in [3]), if ${a_0}^2 = \frac{1-p}{4}$ and ${a_1}^2 = \frac{p-1}{4}$, we have that $x^3 = x$. □

Remark 1. In Proposition 1, if we consider $p = 3$, in (4), it follows that $a_0^2 = 1(\text{mod } 3)$ and $a_1^2 = 2 \ (\text{mod } 3)$, which has no set of solutions for a_0 and a_1 in $\mathbb{Z}_3$. For $p = 2$, in equation (4), it leads to $0a_1^2 = 1$, which is impossible.

Proposition 2. Let $x \in \mathbb{H}/\mathbb{Z}_p$ in the form $x = a_1 i + a_2 j + a_3 k$, where at least two of a_1, a_2, a_3 are non-zero. Then $x^3 = x$ if and only if ${a_1}^2 + {a_2}^2 + {a_3}^2 = p - 1$ and $x^3 = -x$ if and only if ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1 - p$.

Proof. We first find the general solution by using the product of a quaternion, $x^3 = (a_0 + a_1 i + a_2 j + a_3 k)^3$.

$$\begin{aligned} x^3 &= (a_0 + a_1 i + a_2 j + a_3 k)^2(a_0 + a_1 i + a_2 j + a_3 k) \\ &= [({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2) \\ &\quad +(a_0 a_1 + a_1 a_0 + a_2 a_3 - a_3 a_2)i \\ &\quad +(a_0 a_2 + a_2 a_0 + a_1 a_3 - a_3 a_1)j \\ &\quad +(a_0 a_3 + a_3 a_0 + a_1 a_2 - a_2 a_1)k](a_0 + a_1 i + a_2 j + a_3 k) \\ &= [({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2) \\ &\quad +(2a_0 a_1)i + (2a_0 a_2)j + (2a_0 a_3)k](a_0 + a_1 i + a_2 j + a_3 k) \\ &= ({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)a_0 \\ &\quad -(2a_0 a_1)a_1 - (2a_0 a_2)a_2 - (2a_0 a_3)a_3 \\ &\quad +[({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)a_1 \\ &\quad +(2a_0 a_1)a_0 + (2a_0 a_2)a_3 - (2a_0 a_3)a_2]i \\ &\quad +[({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)a_2 \\ &\quad +(2a_0 a_2)a_0 + (2a_0 a_1)a_3 - (2a_0 a_3)a_1]j \\ &\quad +[({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)a3 \\ &\quad +(2a_0 a_3)a_0 + (2a_0 a_1)a_2 - (2a_0 a_2)a_1]k \\ &= a_0({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2 - 2{a_1}^2 - 2{a_2}^2 - 2{a_3}^2) \\ &\quad +a_1({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2 + 2{a_0}^2)i \\ &\quad +a_2({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2 + 2{a_0}^2)j \\ &\quad +a_3({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2 + 2{a_0}^2)k \\ &= a_0({a_0}^2 - 3{a_1}^2 - 3{a_2}^2 - 3{a_3}^2) \\ &\quad +a_1(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)i \\ &\quad +a_2(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)j \\ &\quad +a_3(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)k \end{aligned}$$

Let $x = a_1 i + a_2 j + a_3 k$. Then for $x^3 = x$, by Proposition 2 in [3], we have ${a_1}^2 + {a_2}^2 + {a_3}^2 = p - 1$.

For $x^3 = -x$. We have $(a_1 i + a_2 j + a_3 k)^3 = -(a_1 i + a_2 j + a_3 k)$, hence

$$a_1(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)i + a_2(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)j + a_3(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)k = -a_1 i - a_2 j - a_3 k.$$

We will get the following three equations:

$$a_i(-{a_1}^2 - {a_2}^2 - {a_3}^2) = -a_i\ ;\ i = 1, 2, 3.$$

From the above equations we obtain:

$$a_i = 0 \text{ or } -{a_1}^2 - {a_2}^2 - {a_3}^2 = -1\ ; i = 1, 2, 3.$$

From the equation $-{a_1}^2 - {a_2}^2 - {a_3}^2 = -1$, we have ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1$ and we can also write in ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1 - p$ as $p = 0 \pmod p$.

For the converse, the hypothesis given that ${a_1}^2 + {a_2}^2 + {a_3}^2 = 1 - p$. Hence,

$$\begin{aligned} x^3 = (a_1 i + a_2 j + a_3 k)^3 &= a_1(-{a_1}^2 - {a_2}^2 - {a_3}^2)i \\ &\quad + a_2(-{a_1}^2 - {a_2}^2 - {a_3}^2)j + a_3(-{a_1}^2 - {a_2}^2 - {a_3}^2)k \\ &= a_1(p-1)i + a_2(p-1)j + a_3(p-1)k \\ &= -{a_1}^2 i - {a_2}^2 j - {a_3}^2 k \text{ as } p = 0 \pmod p \\ &= -x. \end{aligned}$$

Therefore, $x^3 = -x$.
If ${a_1}^2 + {a_2}^2 + {a_3}^2 = p - 1$, then by Proposition 2 in [3], thus $x^3 = x$ □

Remark 2. In Proposition 2, if we consider $p = 3$, for the case $x^3 = -x$, it follows that $a_0^2 = 2 \pmod 3$ which has no solutions in $\mathbb{Z}_3$. For $p = 2$, it leads to $0a_1^2 = 1$, which is impossible.

Theorem. Let $x \in \mathbb{H}/\mathbb{Z}_p$ where p is a prime number where $p \neq 2,3$. Let x be an element of the form $x = a_0 + a_1 i + a_2 j + a_3 k$, where $a_0 \neq 0$ and at least one of a_1, a_2, a_3 is non-zero. Then, $x^3 = x$ if and only if ${a_0}^2 = \frac{1-p}{4}$ and ${a_1}^2 + {a_2}^2 + {a_3}^2 = \frac{p-1}{4}$ and $x^3 = -x$ if and only if ${a_0}^2 = \frac{p-1}{4}$ and ${a_1}^2 + {a_2}^2 + {a_3}^2 = \frac{1-p}{4}$.

Proof. Let $x = a_0 + a_1 i + a_2 j + a_3 k$. If $x^3 = x$, then by follow readily by Theorem 1 in [3]. If $x^3 = -x$, then $(a_0 + a_1 i + a_2 j + a_3 k)^3 = -(a_0 + a_1 i + a_2 j + a_3 k)$
From the result in **Proposition 2**, we get:

$$\begin{aligned} &a_0({a_0}^2 - 3{a_1}^2 - 3{a_2}^2 - 3{a_3}^2) + a_1(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)i \\ &+ a_2(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)j + a_3(3{a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2)k \\ &= -a_0 - a_1 i - a_2 j - a_3 k. \end{aligned}$$

Then, we obtain four equations as listed below, by equating the corresponding coefficients:

$$a_0({a_0}^2 - 3({a_1}^2 + {a_2}^2 + {a_3}^2)) = -a_0$$
$$a_i\left(3{a_0}^2 - ({a_1}^2 + {a_2}^2 + {a_3}^2)\right) = -a_i; i = 1, 2, 3.$$

From the above four equations we get:

$$a_0 = 0 \text{ or } {a_0}^2 - 3({a_1}^2 + {a_2}^2 + {a_3}^2) = -1$$
$$a_i = 0 \text{ or } 3{a_0}^2 - ({a_1}^2 + {a_2}^2 + {a_3}^2) = -1\,; i = 1, 2, 3.$$

Since $a_0 \neq 0$, we have ${a_0}^2 - 3({a_1}^2 + {a_2}^2 + {a_3}^2) = -1$ and from the last three equations we have $3{a_0}^2 - ({a_1}^2 + {a_2}^2 + {a_3}^2) = -1$. Let ${a_1}^2 + {a_2}^2 + {a_3}^2 = \lambda$. It follows by two equations:

$$ {a_0}^2 - 3\lambda = -1 \quad (5)$$
$$ 3{a_0}^2 - \lambda = -1 \quad (6).$$

By equating both equations,

$$ {a_0}^2 - 3\lambda = 3{a_0}^2 - \lambda \Rightarrow -2{a_0}^2 = 2\lambda \Rightarrow {a_0}^2 = -\lambda\,.$$

Substitute ${a_0}^2 = -\lambda$ into (5) and we get $-4\lambda = -1 \Rightarrow \lambda = \frac{1}{4} = \frac{1-p}{4}$ as $p = 0 \pmod p$. Hence, ${a_1}^2 + {a_2}^2 + {a_3}^2 = \frac{1-p}{4}$. And since ${a_0}^2 = -\lambda$, we get ${a_0}^2 = \frac{p-1}{4}$.
For the converse, given ${a_0}^2 = \frac{p-1}{4}$ and ${a_1}^2 + {a_2}^2 + {a_3}^2 = \frac{1-p}{4}$. Hence,

$$
\begin{aligned}
x^3 &= (a_0 + a_1 i + a_2 j + a_3 k)^3 \\
&= a_0({a_0}^2 - 3({a_1}^2 + {a_2}^2 + {a_3}^2)) + a_1(3{a_0}^2 - ({a_1}^2 + {a_2}^2 + {a_3}^2))i \\
&\quad + a_2(3{a_0}^2 - ({a_1}^2 + {a_2}^2 + {a_3}^2))j + a_3(3{a_0}^2 - ({a_1}^2 + {a_2}^2 + {a_3}^2))k \\
&= a_0\left(\frac{p-1}{4} - 3\,\frac{1-p}{4}\right) + a_1\left(3\,\frac{p-1}{4} - \frac{1-p}{4}\right)i + a_2\left(3\,\frac{p-1}{4} - \frac{1-p}{4}\right))j + a_3\left(3\,\frac{p-1}{4} - \frac{1-p}{4}\right)k \\
&= a_0(p-1) + a_1(p-1)i + a_2(p-1)j + a_3(p-1)k \\
&= -a_0 - a_1 i - a_2 j - a_3 k, \text{as } p = 0 \pmod p \\
&= -x.
\end{aligned}
$$

Hence, we have $x^3 = -x$. □

3 Conclusion and recommendations

There are some potential applications of idempotents, tripotent, or more broadly k-potent ring elements. Wu [5] defined these k-potent matrices and their variations using the equation $A^k = \lambda I + \mu A$, where $\lambda\mu = 0$, $\lambda, \mu \in \{-1, 0, 1\}$, and $k \geq 2$. In [5], Wu demonstrated the utility of k-potent matrices in digital image encryption. The encryption method involves utilizing a series of encryption key matrices to modify the gray level of each pixels' matrix multiplications, masking the original image and producing a transformed version.

Lastly, we have found out that weakly tripotent elements exist in quaternion rings over $\mathbb{Z}_p$. As the future recommendations, the weakly tripotent with other classes of rings can be further investigated. For example, every element is the sum of two weakly tripotents and a tripotent or each element is the sum of a weakly tripotent, a tripotent and an idempotent.

The authors would like to thank for the financial support provided by Universiti Tunku Abdul Rahman Research Fund, Malaysia (IPSR/RMC/UTARRF/2023-C1/Y01).

References

[1] P. V. Danchev, Kragujev. J. Math. **43**, 465-469 (2019)

[2] M. Aristidou, A. Demetre, Int. J. Algebra, **6(5)**, 249-254 (2012)

[3] M. Aristidou, K. Hailemariam, Acta Univ. Sapientiae Matem. **13**, 78-87 (2021)

[4] W. R. Hamilton, Elements of quaternions (1866)

[5] Y. Wu, American-Math. **10**, 455-460 (2010)