Weakly Tripotent Elements in Quaternion Rings over $\mathbb{Z}_p$

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Abstract. Let $R$ be a ring. An element $x \in R$ is weakly tripotent if $x$ can be written as $x^3 = x$ or $x^3 = -x$. In this paper, we discuss weakly tripotency in quaternion rings over $\mathbb{Z}_p$, $\mathbb{H}/\mathbb{Z}_p$. We also give some conditions for the element $x \in \mathbb{H}/\mathbb{Z}_p$ to be weakly tripotent.

1 Introduction

In the paper of Danchev [1], an element $x$ of ring $R$ is said to be weakly tripotents if $x^3 = x$ or $x^3 = -x$. A ring $R$ is called weakly tripotent if all of its elements are weakly tripotent. In [2], Aristidou and Demetre provide examples and they also establish some conditions for idempotency in $\mathbb{H}/\mathbb{Z}_p$, where $p$ is prime. Besides that, Aristidou and Hailemariam (in [3]) used the similar methods in [2], they able to give some conditions for tripotency of quaternion rings over $\mathbb{Z}_p$ ($p$ is prime). In this paper, we give some conditions for a quaternion ring over $\mathbb{Z}_p$ ($p$ is prime) to be weakly tripotent.

2 Weakly tripotent elements $\mathbb{H}/\mathbb{Z}_p$

A set of real quaternions, we denote it as $\mathbb{H}$, was first introduced by Hamilton [4] in 1866 as an extension of complex number into four dimensions. A quaternion can be written as the form $x = x_0 + x_1 i + x_2 j + x_3 k$ where $x_n$ are reals and $i, j, k$ are complex elements such that $i^2 = j^2 = k^2 = ij = jk = -1$. Note that $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$.

When multiplying a pair of quaternions $x = x_0 + x_1 i + x_2 j + x_3 k$ and $y = y_0 + y_1 i + y_2 j + y_3 k$, we can compute the product $xy$ as

$$xy = (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3) + (x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2)i + (x_0 y_2 + x_2 y_0 + x_1 y_3 - x_3 y_1)j + (x_0 y_3 + x_3 y_0 + x_1 y_2 - x_2 y_1)k.$$ 

For the case, $x \in \mathbb{H}$ and $x = a_0 + a_1 i + a_2 j + a_3 k$, where $a_0 \neq 0$, $a_1 = a_2 = a_3 = 0$. The only weakly tripotent elements are $x = -1, 0$ and $1$. We are focus on whether some weakly

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tripotent elements are in the quaternion rings over \( \mathbb{Z}_p \). We first give the following two propositions and prove that the element \( x \) is weakly tripotent if the conditions stated in the proposition have been satisfied.

**Proposition 1.** Let \( x \in \mathbb{H}/\mathbb{Z}_p \) be a quaternion element of the form \( x = a_0 + a_1 i \), where \( a_0 \) and \( a_1 \) are nonzeros. Then \( x^3 = x \) if and only if \( a_0^2 = \frac{1+p}{4} \) and \( a_1^2 = \frac{p-1}{4} \) and \( x^3 = -x \) if and only if \( a_0^2 = \frac{p-1}{4} \) and \( a_1^2 = \frac{1+p}{4} \), where \( p \) is prime and \( p \neq 2, 3 \).

**Proof.** Let \( x = a_0 + a_1 i \) and \( x \) is weakly tripotent in \( \mathbb{H}/\mathbb{Z}_p \). Then \( x^3 = x \) or \( x^3 = -x \).

For \( x^3 = x \), by Proposition 1 in [3], it follows that \( a_0^2 = \frac{1+p}{4} \) and \( a_1^2 = \frac{p-1}{4} \).

For \( x^3 = x \), we have, \((a_0 + a_1 i)^3 = -(a_0 + a_1 i)\). This follows that \( a_0^3 - 3a_0a_1^2 + (3a_0^2a_1 - a_1^3)i = -a_0 - a_1 i \). Thus, we have these two equations:

\[
\begin{align*}
a_0^3 - 3a_0a_1^2 &= -a_0 \\
3a_0^2a_1 - a_1^3 &= -a_1 .
\end{align*}
\]

Simplify the above equations, we will obtain

\[
\begin{align*}
a_0^2 - 3a_1^2 &= -1 \quad (1) \\
a_0^2 - a_1^2 &= -1 \quad (2)
\end{align*}
\]

By solving the equations (1) and (2), we have

\[
a_0^2 - 3a_1^2 = 3a_0^2 - a_1^2 \Rightarrow -2a_0^2 = 2a_1^2 \Rightarrow a_0^2 = -a_1^2 \quad \text{------(3).}
\]

By substitute \( a_0^2 = -a_1^2 \) into (1) and we get

\[
-a_1^2 - 3a_1^2 = -1 \Rightarrow -4a_1^2 = -1 \Rightarrow a_1^2 = \frac{1}{4} \quad \text{------(4).}
\]

Since \( p = 0 \pmod{p} \), \( a_1^2 = \frac{1-p}{4} \). Substitute \( a_1^2 \) into (3) and we get \( a_0^2 = 4^{-1}(p - 1) \).

To see if the quantities \( \frac{1-p}{4} \) and \( \frac{p-1}{4} \) are squares mod \( p \), we calculate the Legendre Symbol for \( \frac{1-p}{4} \) and \( \frac{p-1}{4} \) respectively. The first gives:

\[
\left( \frac{1-p}{4} \right) = 1
\]

and the second gives:

\[
\left( \frac{p-1}{4} \right) = \left( \frac{p-1}{p} \right) \left( \frac{1}{p} \right) = \left( \frac{p-1}{p} \right) \cdot 1 = (p - 1) \frac{p-1}{2} = (-1) \frac{p-1}{2} = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{4} \\ -1, & \text{if } p \equiv 3 \pmod{4} \end{cases}
\]

Hence, there are no weakly tripotents of the form \( a_0 + a_1 i \), if \( p \equiv 3 \pmod{4} \). It follows that if weakly tripotent element exists in the form of \( a_0 + a_1 i \) with \( a_0 \) and \( a_1 \) both nonzero,
then \( p \equiv 1 \text{(mod 4)} \) and in this case, \( a_0^2 = \frac{p-1}{4} \) and \( a_1^2 = \frac{1-p}{4} \). Therefore, existence of weakly tripotent elements may classify the nature of \( p \). For the converse, given \( a_0^2 = \frac{p-1}{4} \) and \( a_1^2 = \frac{1-p}{4} \), we have that:

\[
x^3 = (a_0 + a_1 i)^3 = a_0^3 - 3a_0a_1^2 + (3a_0^2a_1 - a_1^3)i
\]

\[
= a_0(a_0^2 - 3a_1^2) + a_1(3a_0^2 - a_1^2)i
\]

\[
= a_0(\frac{p-1}{4} - 3 \frac{1-p}{4}) + a_1\left(3 \frac{p-1}{4} - \frac{1-p}{4}\right)i
\]

\[
= a_0(p-1) + a_1(p-1)i
\]

\[
= -a_0 - a_1 i, \text{as } p = 0 \text{ (mod } p)\]

\[
= -x.
\]

Hence, \( x \) is weakly tripotent. Similarly, in Proposition 1 (in [3]), if \( a_0^2 = \frac{1-p}{4} \) and \( a_1^2 = \frac{p-1}{4} \), we have that \( x^3 = x \).

**Remark 1.** In Proposition 1, if we consider \( p = 3 \), in (4), it follows that \( a_0^2 = 1 \text{(mod 3)} \) and \( a_1^2 = 2 \text{ (mod 3)} \), which has no set of solutions for \( a_0 \) and \( a_1 \) in \( \mathbb{Z}_3 \). For \( p = 2 \), in equation (4), it leads to \( 0a_1^2 = 1 \), which is impossible.

**Proposition 2.** Let \( x \in \mathbb{H}/\mathbb{Z}_p \) in the form \( x = a_1 i + a_2 j + a_3 k \), where at least two of \( a_1, a_2, a_3 \) are non-zero. Then \( x^3 = x \) if and only if \( a_1^2 + a_2^2 + a_3^2 = p - 1 \) and \( x^3 = -x \) if and only if \( a_1^2 + a_2^2 + a_3^2 = 1 - p \).

**Proof.** We first find the general solution by using the product of a quaternion, \( x^3 = (a_0 + a_1 i + a_2 j + a_3 k)^3 \).

\[
x^3 = (a_0 + a_1 i + a_2 j + a_3 k)^2(a_0 + a_1 i + a_2 j + a_3 k)
\]

\[
= (a_0^2 - a_1^2 - a_2^2 - a_3^2)
\]

\[
+ (a_0a_1 + a_1 a_0 + a_2 a_3 - a_3 a_2)i
\]

\[
+ (a_0 a_2 + a_2 a_0 + a_1 a_3 - a_3 a_1)j
\]

\[
+ (a_0 a_3 + a_3 a_0 + a_1 a_2 - a_2 a_1)k(a_0 + a_1 i + a_2 j + a_3 k)
\]

\[
= (a_0^2 - a_1^2 - a_2^2 - a_3^2)
\]

\[
+ (2a_0 a_1)i + (2a_0 a_2)j + (2a_0 a_3)k(a_0 + a_1 i + a_2 j + a_3 k)
\]

\[
= (a_0^2 - a_1^2 - a_2^2 - a_3^2)a_0
\]

\[
- (2a_0 a_1)a_1 - (2a_0 a_2)a_2 - (2a_0 a_3)a_3
\]

\[
+ [(a_0^2 - a_1^2 - a_2^2 - a_3^2)a_1)
\]

\[
+ [(2a_0 a_1)a_0 + (2a_0 a_2)a_3 - (2a_0 a_3)a_2]i
\]

\[
+ [(a_0^2 - a_1^2 - a_2^2 - a_3^2)a_2)
\]

\[
+ (2a_0 a_2)a_0 + (2a_0 a_1)a_3 - (2a_0 a_3)a_1]j
\]

\[
+ [(a_0^2 - a_1^2 - a_2^2 - a_3^2)a_3)
\]

\[
+ (2a_0 a_3)a_0 + (2a_0 a_1)a_2 - (2a_0 a_2)a_1]k
\]

\[
= a_0(a_0^2 - a_1^2 - a_2^2 - a_3^2 - 2a_2^2 - 2a_3^2)
\]

\[
+ a_1(a_0^2 - a_1^2 - a_2^2 - a_3^2 + 2a_0^2)i
\]

\[
+ a_2(a_0^2 - a_1^2 - a_2^2 - a_3^2 + 2a_0^2)j
\]

\[
+ a_3(a_0^2 - a_1^2 - a_2^2 - a_3^2 + 2a_0^2)k
\]

\[
= a_0(a_0^2 - 3a_1^2 - 3a_2^2 - 3a_3^2)
\]

\[
+ a_1(3a_0^2 - a_1^2 - a_2^2 - a_3^2)i
\]

\[
+ a_2(3a_0^2 - a_1^2 - a_2^2 - a_3^2)j
\]

\[
+ a_3(3a_0^2 - a_1^2 - a_2^2 - a_3^2)k
\]
Let \( x = a_1i + a_2j + a_3k \). Then for \( x^3 = x \), by Proposition 2 in [3], we have \( a_1^2 + a_2^2 + a_3^2 = p - 1 \).

For \( x^3 = -x \). We have \((a_1i + a_2j + a_3k)^3 = -(a_1i + a_2j + a_3k)\), hence
\[
\begin{align*}
a_1(3a_0^2 - a_1^2 - a_2^2 - a_3^2)i + a_2(3a_0^2 - a_1^2 - a_2^2 - a_3^2)j \\
+ a_3(3a_0^2 - a_1^2 - a_2^2 - a_3^2)k &= -a_1i - a_2j - a_3k.
\end{align*}
\]

We will get the following three equations:
\[
\begin{align*}
a_i(-a_i^2 - a_i^2 - a_i^2) &= -a_i; \ i = 1, 2, 3.
\end{align*}
\]

From the above equations we obtain:
\[
\begin{align*}
a_i &= 0 \text{ or } -a_i^2 - a_i^2 - a_i^2 = -1; \ i = 1, 2, 3.
\end{align*}
\]

From the equation \(-a_i^2 - a_i^2 - a_i^2 = -1\), we have \(a_1^2 + a_2^2 + a_3^2 = 1\) and we can also write in \(a_1^2 + a_2^2 + a_3^2 = 1 - p\) as \(p = 0 \mod p\).

For the converse, the hypothesis given that \(a_1^2 + a_2^2 + a_3^2 = 1 - p\). Hence,
\[
\begin{align*}
x^3 &= (a_1i + a_2j + a_3k)^3 \\
&= a_1(-a_1^2 - a_2^2 - a_3^2)i \\
&\quad + a_2(-a_1^2 - a_2^2 - a_3^2)j + a_3(-a_1^2 - a_2^2 - a_3^2)k \\
&= a_4(p - 1)i + a_2(p - 1)j + a_3(p - 1)k \\
&= -a_1^2i - a_2^2j - a_3^2k \text{ as } p = 0 \mod p \\
&= -x.
\end{align*}
\]

Therefore, \(x^3 = -x\).

If \(a_1^2 + a_2^2 + a_3^2 = p - 1\), then by Proposition 2 in [3], thus \(x^3 = x\) \(\square\)

**Remark 2.** In Proposition 2, if we consider \(p = 3\), for the case \(x^3 = -x\), it follows that \(a_0^2 = 2 \mod 3\) which has no solutions in \(\mathbb{Z}_3\). For \(p = 2\), it leads to \(0a_1^2 = 1\), which is impossible.

**Theorem.** Let \(x \in \mathbb{H}/\mathbb{Z}_p\) where \(p\) is a prime number where \(p \neq 2, 3\). Let \(x\) be an element of the form \(x = a_0 + a_1i + a_2j + a_3k\), where \(a_0 \neq 0\) and at least one of \(a_1, a_2, a_3\) is non-zero. Then, \(x^3 = x\) if and only if \(a_0^2 = \frac{1-p}{4}\) and \(a_1^2 + a_2^2 + a_3^2 = \frac{p-1}{4}\) and \(x^3 = -x\) if and only if \(a_0^2 = \frac{p-1}{4}\) and \(a_1^2 + a_2^2 + a_3^2 = \frac{1-p}{4}\).

**Proof.** Let \(x = a_0 + a_1i + a_2j + a_3k\). If \(x^3 = x\), then by follow readily by Theorem 1 in [3]. If \(x^3 = -x\), then \((a_0 + a_1i + a_2j + a_3k)^3 = -(a_0 + a_1i + a_2j + a_3k)\)

From the result in Proposition 2, we get:
\[
\begin{align*}
a_0(a_0^2 - 3a_1^2 - 3a_2^2 - 3a_3^2) + a_1(3a_0^2 - a_1^2 - a_2^2 - a_3^2)i \\
+ a_2(3a_0^2 - a_1^2 - a_2^2 - a_3^2)j + a_3(3a_0^2 - a_1^2 - a_2^2 - a_3^2)k \\
&= -a_0 - a_1i - a_2j - a_3k.
\end{align*}
\]

Then, we obtain four equations as listed below, by equating the corresponding coefficients:
\[
\begin{align*}
a_0(a_0^2 - 3(a_1^2 + a_2^2 + a_3^2)) &= -a_0 \\
a_i(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)) &= -a_i; \ i = 1, 2, 3.
\end{align*}
\]
From the above four equations we get:

\[ a_0 = 0 \text{ or } a_0^2 - 3(a_1^2 + a_2^2 + a_3^2) = -1 \]
\[ a_i = 0 \text{ or } 3a_0^2 - (a_i^2 + a_2^2 + a_3^2) = -1; \ i = 1, 2, 3. \]

Since \( a_0 \neq 0 \), we have \( a_0^2 - 3(a_1^2 + a_2^2 + a_3^2) = -1 \) and from the last three equations we have \( 3a_0^2 - (a_1^2 + a_2^2 + a_3^2) = -1 \). Let \( a_1^2 + a_2^2 + a_3^2 = \lambda \). It follows by two equations:

\[ a_0^2 - 3\lambda = -1 \quad (5) \]
\[ 3a_0^2 - \lambda = -1 \quad (6). \]

By equating both equations,

\[ a_0^2 - 3\lambda = 3a_0^2 - \lambda \Rightarrow -2a_0^2 = 2\lambda \Rightarrow a_0^2 = -\lambda. \]

Substitute \( a_0^2 = -\lambda \) into (5) and we get \(-4\lambda = -1 \Rightarrow \lambda = \frac{1}{4} = \frac{1-p}{4} \) as \( p = 0 \) (mod \( p \)).

Hence, \( a_1^2 + a_2^2 + a_3^2 = \frac{1-p}{4} \). And since \( a_0^2 = -\lambda \), we get \( a_0^2 = \frac{p-1}{4} \).

For the converse, given \( a_0^2 = \frac{p-1}{4} \) and \( a_1^2 + a_2^2 + a_3^2 = \frac{1-p}{4} \). Hence,

\[ x^3 = (a_0 + a_1 i + a_2 j + a_3 k)^3 \]
\[ = a_0(3a_0^2 - 3(a_1^2 + a_2^2 + a_3^2)) + a_1(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)) i \]
\[ + a_2(3a_0^2 - (a_2^2 + a_2^2 + a_3^2)) j + a_3(3a_0^2 - (a_1^2 + a_2^2 + a_3^2)) k \]
\[ = a_0 \left( \frac{p-1}{4} - 1 \right) i + a_1 \left( 3 \frac{p-1}{4} - \frac{1-p}{4} \right) i + a_2 \left( 3 \frac{p-1}{4} - \frac{1-p}{4} \right) j + a_3 \left( 3 \frac{p-1}{4} - \frac{1-p}{4} \right) k \]
\[ = a_0(p - 1) + a_1(p - 1)i + a_2(p - 1)j + a_3(p - 1)k \]
\[ = -a_0 - a_1 i - a_2 j - a_3 k, \text{ as } p = 0 \text{ (mod } p) \]
\[ = -x. \]

Hence, we have \( x^3 = -x \). \( \square \)

### 3 Conclusion and recommendations

There are some potential applications of idempotents, tripotent, or more broadly \( k \)-potent ring elements. Wu [5] defined these \( k \)-potent matrices and their variations using the equation \( A^k = \lambda I + \mu A \), where \( \lambda, \mu = 0 \), \( \lambda, \mu \in \{-1, 0, 1\} \), and \( k \geq 2 \). In [5], Wu demonstrated the utility of \( k \)-potent matrices in digital image encryption. The encryption method involves utilizing a series of encryption key matrices to modify the gray level of each pixels’ matrix multiplications, masking the original image and producing a transformed version.

Lastly, we have found out that weakly tripotent elements exist in quaternion rings over \( \mathbb{Z}_p \). As the future recommendations, the weakly tripotent with other classes of rings can be further investigated. For example, every element is the sum of two weakly tripotents and a tripotent or each element is the sum of a weakly tripotent, a tripotent and an idempotent.

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