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Some Properties of Hyperbolic k-Narayana Quaternions

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Abstract. In this paper, we introduced the hyperbolic k-Narayana quaternions. Several properties of these quaternions are investigated, including the Binet formulas, generating functions, and summation formulas. Moreover, some identities such as Catalan, Cassini, and d'Ocagne are obtained. Our results extend and generalize well-known theorems.

1 Introduction

Researchers have been studying number sequences for a considerable amount of time. Specifically, the Fibonacci numbers are regarded as the most significant and noteworthy among these numbers. Over the years, extensive research has been conducted on Fibonacci numbers, leading to numerous generalizations. Examples of these generalizations include the k-Fibonacci, (p, q)-Fibonacci, and Jacobsthal numbers.

There exists a sequence of numbers that defines a recursive relationship similar to the Fibonacci numbers, known as the Narayana numbers. In the 14th century, Tadepalli Venkata Narayana introduced them as follows:

$$\mathfrak{N}_n = \mathfrak{N}_{n-1} + \mathfrak{N}_{n-3}, \quad n \geq 3 \tag{1}$$

with 0,1,1,1 initial condition.
That is, the Narayana sequence is $\{0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, \dots\}$.

Narayana numbers provide a powerful tool for counting and analyzing combinatorial structures, and their applications extend to diverse areas within mathematics and computer science (see [1, 3, 15]).

To generalize the definition of the Narayana numbers, the k-Narayana numbers were introduced in 2015 by Ramírez and Sirvent [14].

Definition 1.1. For a non-zero integer number k, the k-Narayana sequence is given by

$$\mathfrak{N}_{k,n} = k\mathfrak{N}_{k,n-1} + \mathfrak{N}_{k,n-3}, \quad n \geq 3 \tag{2}$$

with $0, 1, k, k^2$ initial condition.

The equation $x^3 + kx^2 - 1 = 0$ corresponds to the characteristic equation of the recurrence relation (2), with roots denoted as $\varphi_k, \varpi_k, \varsigma_k$. The following equation represents Binet's formula for the k-Narayana numbers [14]:

$$\mathfrak{N}_{k,n} = \frac{\varphi_k^{n+1}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+1}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+1}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}, \quad n \geq 0 \tag{3}$$

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Özkan [12] has illustrated the summation formulas for k-Narayana numbers as follows:

$$\sum_{i=0}^{n} \mathfrak{N}_{k,i} = \frac{\mathfrak{N}_{k,1} + \mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1} + \mathfrak{N}_{k,n+2}}{k} - \mathfrak{N}_{k,n+1}. \tag{4}$$

Mathematician Sir William Rowan Hamilton introduced quaternions [8] for the first time in 1843. Hamilton developed quaternions as a way to extend complex numbers to higher dimensions, particularly to three-dimensional space. He was inspired by the complex numbers, which can be thought of as points on a plane, and sought a similar system for representing points in three-dimensional space. This led him to the invention of quaternions, which have applications in various fields, including computer graphics, robotics, and signal processing (see [2, 4]). Quaternions with specific integer components have been the subject of numerous investigations. The Fibonacci and Lucas quaternions were first introduced and some properties were provided by Horadam [9] in 1963. Later, Çimen and İpek [5] studied Pell quaternions and Pell-Lucas quaternions and provided insights into their properties, applications, and implications in various mathematical fields.

In 1902, Macfarlane [11] introduced the hyperbolic quaternions which, unlike real quaternions, do not exhibit commutativity, and conducted a study on their properties. Hyperbolic quaternions find applications in various fields, including physics, computer graphics, and geometric algebra (see [6, 16]). A set of hyperbolic quaternions are represented as

$$H = \{h = o + p\boldsymbol{j_1} + x\boldsymbol{j_2} + y\boldsymbol{j_3} : \quad o, p, x, y \in \mathbb{R}\}$$

where $\boldsymbol{j_1}, \boldsymbol{j_2}$ and $\boldsymbol{j_3}$ are hyperbolic quaternion units satisfying the conditions

$$\boldsymbol{j_1}^2 = \boldsymbol{j_2}^2 = \boldsymbol{j_3}^2 = \boldsymbol{j_1 j_2 j_3} = 1,\ \boldsymbol{j_1 j_2} = \boldsymbol{j_3} = -\boldsymbol{j_2 j_1},\ \boldsymbol{j_2 j_3} = \boldsymbol{j_1} = -\boldsymbol{j_3 j_2}, \boldsymbol{j_3 j_1} = \boldsymbol{j_2} = -\boldsymbol{j_1 j_3}.$$

Let $\mathbf{h}_1 = o_1 + p_1\boldsymbol{j_1} + x_1\boldsymbol{j_2} + y_1\boldsymbol{j_3}$ and $\mathbf{h}_2 = o_2 + p_2\boldsymbol{j_1} + x_2\boldsymbol{j_2} + y_2\boldsymbol{j_3}$ denote two hyperbolic quaternions. Equality, addition, subtraction, and scalar multiplication and multiplication can be defined as follows:

$$\begin{aligned}
\mathbf{h}_1 = \mathbf{h}_2 &\text{ only if } \ o_1 = o_2, p_1 = p_2, x_1 = x_2, y_1 = y_2;\\
\mathbf{h}_1 + \mathbf{h}_2 &= (o_1 + o_2) + (p_1 + p_2)\boldsymbol{j_1} + (x_1 + x_2)\boldsymbol{j_2} + (y_1 + y_2)\boldsymbol{j_3};\\
\mathbf{h}_1 - \mathbf{h}_2 &= (o_1 - o_2) + (p_1 - p_2)\boldsymbol{j_1} + (x_1 - x_2)\boldsymbol{j_2} + (y_1 - y_2)\boldsymbol{j_3};\\
\lambda\mathbf{h}_1 &= \lambda o_1 + \lambda p_1\boldsymbol{j_1} + \lambda x_1\boldsymbol{j_2} + \lambda y_1\boldsymbol{j_3}, \quad \lambda \in \mathbb{R}.\\
\mathbf{h}_1\mathbf{h}_2 &= (o_1o_2 + p_1p_2 + x_1x_2 + y_1y_2) + (o_1p_2 + p_1o_2 + x_1y_2 - y_1x_2)\boldsymbol{j_1}\\
&\quad + (o_1x_2 - p_1y_2 + x_1o_2 + y_1p_2)\boldsymbol{j_2} + (o_1y_2 + p_1x_2 - x_1p_2 + y_1o_2)\boldsymbol{j_3}.
\end{aligned}$$

Then, the set H of all hyperbolic quaternions is a vector space over a field $\mathbb{R}$. Moreover, the conjugate of $h = o + p\boldsymbol{j_1} + x\boldsymbol{j_2} + y\boldsymbol{j_3}$ is established by

$$\bar{h} = o - p\boldsymbol{j_1} - x\boldsymbol{j_2} - y\boldsymbol{j_3}.$$

Many researchers have defined and studied various types of hyperbolic quaternions, including hyperbolic k-Fibonacci, hyperbolic k-Fibonacci-Lucas, hyperbolic k-Jacobsthal, and hyperbolic k-Jacobsthal-Lucas quaternions (see [7, 10, 13]).

Several authors worked on the Catalan, Cassini, and d'Ocagne identities because they are useful in various mathematical contexts, including combinatorics, probability theory, matrix theory, coding theory, and linear algebra.

The objective of this study is to present the hyperbolic k-Narayana quaternions, including their generating function, exponential generating function, and Binet's formula. Additionally, we derive the Catalan, Cassini, and d'Ocagne identities for the hyperbolic k-Narayana quaternions using Binet's formula.

2 Main Results

In this part, we will initially provide the definition of the hyperbolic k-Narayana quaternions. Subsequently, we will explore various properties associated with these quaternions.

Definition 2.1. Assume $n \geq 0$. The nth $k-$Narayana hyperbolic quaternions, $H\mathfrak{N}_{k,n}$, are defined by

$$H\mathfrak{N}_{k,n} = \mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1}\boldsymbol{j_1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_2} + \mathfrak{N}_{k,n+3}\boldsymbol{j_3} \tag{5}$$

where $\boldsymbol{j_1}, \boldsymbol{j_2}, \boldsymbol{j_3}$ satisfy equalities

$$\boldsymbol{j_1^2} = \boldsymbol{j_2^2} = \boldsymbol{j_3^2} = \boldsymbol{j_1 j_2 j_3} = 1, \boldsymbol{j_1 j_2} = \boldsymbol{j_3} = -\boldsymbol{j_2 j_1}, \boldsymbol{j_2 j_3} = \boldsymbol{j_1} = -\boldsymbol{j_3 j_2}, \boldsymbol{j_3 j_1} = \boldsymbol{j_2} = -\boldsymbol{j_1 j_3}.$$

The first few terms of hyperbolic k-Narayana quaternions are given by

$$\begin{aligned}
H\mathfrak{N}_{k,0} &= \boldsymbol{j}_1 + k\boldsymbol{j}_2 + k^2\boldsymbol{j}_3,\\
H\mathfrak{N}_{k,1} &= 1 + k\boldsymbol{j}_1 + k^2\boldsymbol{j}_2 + (k^3+1)\boldsymbol{j}_3,\\
H\mathfrak{N}_{k,2} &= k + k^2\boldsymbol{j}_1 + (k^3+1)\boldsymbol{j}_2 + (k^4+2k)\boldsymbol{j}_3,\\
H\mathfrak{N}_{k,3} &= k^2 + (k^3+1)\boldsymbol{j}_1 + (k^4+2k)\boldsymbol{j}_2 + (k^5+3k^2)\boldsymbol{j}_3,\\
H\mathfrak{N}_{k,4} &= (k^3+1) + (k^4+2k)\boldsymbol{j}_1 + (k^5+3k^2)\boldsymbol{j}_2 + (k^6+4k^3+1)\boldsymbol{j}_3,\\
&\vdots
\end{aligned}$$

Next, we present the recurrence relations of k-Narayana hyperbolic quaternions.

Lemma 2.2. *Assume $n \geq 3$, then*

$$H\mathfrak{N}_{k,n} = kH\mathfrak{N}_{k,n-1} + H\mathfrak{N}_{k,n-3}. \tag{6}$$

Proof. Using (5) and (2), we obtain

$$\begin{aligned}
H\mathfrak{N}_{k,n} &= \mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1}\boldsymbol{j_1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_2} + \mathfrak{N}_{k,n+3}\boldsymbol{j_3}\\
&= (k\mathfrak{N}_{k,n-1} + \mathfrak{N}_{k,n-3}) + (k\mathfrak{N}_{k,n} + \mathfrak{N}_{k,n-2})\boldsymbol{j_1} + (k\mathfrak{N}_{k,n+1} + \mathfrak{N}_{k,n-1})\boldsymbol{j_2}\\
&\quad + (k\mathfrak{N}_{k,n+2} + \mathfrak{N}_{k,n})\boldsymbol{j_3}\\
&= (k\mathfrak{N}_{k,n-1} + k\mathfrak{N}_{k,n}\boldsymbol{j_1} + k\mathfrak{N}_{k,n+1}\boldsymbol{j_2} + k\mathfrak{N}_{k,n+2}\boldsymbol{j_3})\\
&\quad + (\mathfrak{N}_{k,n-3} + \mathfrak{N}_{k,n-2}\boldsymbol{j_1} + \mathfrak{N}_{k,n-1}\boldsymbol{j_2} + \mathfrak{N}_{k,n}\boldsymbol{j_3})\\
&= k(\mathfrak{N}_{k,n-1} + \mathfrak{N}_{k,n}\boldsymbol{j_1} + \mathfrak{N}_{k,n+1}\boldsymbol{j_2} + \mathfrak{N}_{k,n+2}\boldsymbol{j_3})\\
&\quad + (\mathfrak{N}_{k,n-3} + \mathfrak{N}_{k,n-2}\boldsymbol{j_1} + \mathfrak{N}_{k,n-1}\boldsymbol{j_2} + \mathfrak{N}_{k,n}\boldsymbol{j_3})\\
&= kH\mathfrak{N}_{k,n-1} + H\mathfrak{N}_{k,n-3}.
\end{aligned}$$

□

The next theorem shows the relationship between hyperbolic quaternions and their conjugates.

Lemma 2.3. *Assume $n \geq 0$, the following equalities are valid:*

$$H\mathfrak{N}_{k,n} + \overline{H\mathfrak{N}_{k,n}} = 2\mathfrak{N}_{k,n}, \tag{7}$$

$$H\mathfrak{N}_{k,n} - \overline{H\mathfrak{N}_{k,n}} = 2\left(\mathfrak{N}_{k,n+1} + \mathfrak{N}_{k,n+2} + \mathfrak{N}_{k,n+3}\right), \tag{8}$$

$$H\mathfrak{N}_{k,n}\overline{H\mathfrak{N}_{k,n}} = \mathfrak{N}_{k,n}^2 - \mathfrak{N}_{k,n+1}^2 - \mathfrak{N}_{k,n+2}^2 - \mathfrak{N}_{k,n+3}^2. \tag{9}$$

Proof. It is evident that equations (7)-(8) hold. Next, we will demonstrate the validity of equation (9). By (5), we have

$$
\begin{aligned}
&H\mathfrak{N}_{k,n}\overline{H\mathfrak{N}_{k,n}}\\
&= (\mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1}\boldsymbol{j_1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_2} + \mathfrak{N}_{k,n+3}\boldsymbol{j_3})(\mathfrak{N}_{k,n} - \mathfrak{N}_{k,n+1}\boldsymbol{j_1} - \mathfrak{N}_{k,n+2}\boldsymbol{j_2} - \mathfrak{N}_{k,n+3}\boldsymbol{j_3})\\
&= \mathfrak{N}_{k,n}\mathfrak{N}_{k,n} - \mathfrak{N}_{k,n}\mathfrak{N}_{k,n+1}\boldsymbol{j_1} - \mathfrak{N}_{k,n}\mathfrak{N}_{k,n+2}\boldsymbol{j_2} - \mathfrak{N}_{k,n}\mathfrak{N}_{k,n+3}\boldsymbol{j_3}\\
&\quad + \mathfrak{N}_{k,n}\mathfrak{N}_{k,n+1}\boldsymbol{j_1} - \mathfrak{N}_{k,n+1}\mathfrak{N}_{k,n+1}\boldsymbol{j_1}\boldsymbol{j_1} - \mathfrak{N}_{k,n+1}\mathfrak{N}_{k,n+2}\boldsymbol{j_1}\boldsymbol{j_2} - \mathfrak{N}_{k,n+1}\mathfrak{N}_{k,n+3}\boldsymbol{j_1}\boldsymbol{j_3}\\
&\quad + \mathfrak{N}_{k,n}\mathfrak{N}_{k,n+2}\boldsymbol{j_2} - \mathfrak{N}_{k,n+2}\mathfrak{N}_{k,n+1}\boldsymbol{j_2}\boldsymbol{j_1} - \mathfrak{N}_{k,n+2}\mathfrak{N}_{k,n+2}\boldsymbol{j_2}\boldsymbol{j_2} - \mathfrak{N}_{k,n+2}\mathfrak{N}_{k,n+3}\boldsymbol{j_2}\boldsymbol{j_3}\\
&\quad + \mathfrak{N}_{k,n}\mathfrak{N}_{k,n+3}\boldsymbol{j_3} - \mathfrak{N}_{k,n+3}\mathfrak{N}_{k,n+1}\boldsymbol{j_3}\boldsymbol{j_1} - \mathfrak{N}_{k,n+3}\mathfrak{N}_{k,n+2}\boldsymbol{j_3}\boldsymbol{j_2} - \mathfrak{N}_{k,n+3}\mathfrak{N}_{k,n+3}\boldsymbol{j_3}\boldsymbol{j_3}\\
&= \mathfrak{N}_{k,n}^2 - \mathfrak{N}_{k,n+1}^2 - \mathfrak{N}_{k,n+2}^2 - \mathfrak{N}_{k,n+3}^2.
\end{aligned}
$$

□

Theorem 2.4. *(Binet formula for k-Narayana hyperbolic quaternions) Let $\varphi_k, \varpi_k, \varsigma_k$ be the roots of $x^3 + kx^2 - 1 = 0$. Assume $n \geq 0$, then*

$$H\mathfrak{N}_{k,n} = \frac{\varphi_k^{n+1}\hat{\varphi}_k}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+1}\hat{\varpi}_k}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+1}\hat{\varsigma}_k}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}, \tag{10}$$

where $\hat{\varphi}_k = 1+\varphi_k\boldsymbol{j}_1+\varphi_k^2\boldsymbol{j}_2+\varphi_k^3\boldsymbol{j}_3$, $\hat{\varpi}_k = 1+\varpi_k\boldsymbol{j}_1+\varpi_k^2\boldsymbol{j}_2+\varpi_k^3\boldsymbol{j}_3$, and $\hat{\varsigma}_k = 1+\varsigma_k\boldsymbol{j}_1+\varsigma_k^2\boldsymbol{j}_2+\varsigma_k^3\boldsymbol{j}_3$.

Proof. By using (5) and (3) , we have

$$
\begin{aligned}
&H\mathfrak{N}_{k,n} = \mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1}\boldsymbol{j_1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_2} + \mathfrak{N}_{k,n+3}\boldsymbol{j_3}\\
&= \frac{\varphi_k^{n+1}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+1}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+1}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\\
&\quad + \left(\frac{\varphi_k^{n+2}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+2}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+2}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\right)\boldsymbol{j_1}\\
&\quad + \left(\frac{\varphi_k^{n+3}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+3}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+3}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\right)\boldsymbol{j_2}\\
&\quad + \left(\frac{\varphi_k^{n+4}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+4}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+4}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\right)\boldsymbol{j_3}\\
&= \left(\frac{\varphi_k^{n+1}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varphi_k^{n+2}\boldsymbol{j_1}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varphi_k^{n+3}\boldsymbol{j_2}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varphi_k^{n+4}\boldsymbol{j_3}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)}\right)\\
&\quad + \left(\frac{\varpi_k^{n+1}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varpi_k^{n+2}\boldsymbol{j_1}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varpi_k^{n+3}\boldsymbol{j_2}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varpi_k^{n+4}\boldsymbol{j_3}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)}\right)\\
&\quad + \left(\frac{\varsigma_k^{n+1}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)} + \frac{\varsigma_k^{n+2}\boldsymbol{j_1}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)} + \frac{\varsigma_k^{n+3}\boldsymbol{j_2}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)} + \frac{\varsigma_k^{n+4}\boldsymbol{j_3}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\right)\\
&= \frac{\varphi_k^{n+1}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)}\left(1 + \varphi_k\boldsymbol{j_1} + \varphi_k^2\boldsymbol{j_2} + \varphi_k^3\boldsymbol{j_3}\right) + \frac{\varpi_k^{n+1}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)}\left(1 + \varpi_k\boldsymbol{j_1} + \varpi_k^2\boldsymbol{j_2} + \varpi_k^3\boldsymbol{j_3}\right)\\
&\quad + \frac{\varsigma_k^{n+1}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\left(1 + \varsigma_k\boldsymbol{j_1} + \varsigma_k^2\boldsymbol{j_2} + \varsigma_k^3\boldsymbol{j_3}\right)\\
&= \frac{\varphi_k^{n+1}\hat{\varphi}_k}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+1}\hat{\varpi}_k}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+1}\hat{\varsigma}_k}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}.
\end{aligned}
$$

□

Theorem 2.5. *The summation for the $k-$Narayana hyperbolic quaternions is*

$$\sum_{m=0}^{n} H\mathfrak{N}_{k,m} = \frac{H\mathfrak{N}_{k,n} + H\mathfrak{N}_{k,n+1} + H\mathfrak{N}_{k,n+2} - kH\mathfrak{N}_{k,n+1} + 1 + \boldsymbol{j_1} + \boldsymbol{j_2} + \boldsymbol{j_3}}{k} - \boldsymbol{j_2} - (k+1)\boldsymbol{j_3}. \quad (11)$$

Proof. By using (5) and (4), we have

$$\begin{aligned}
\sum_{m=0}^{n} H\mathfrak{N}_{k,m} &= H\mathfrak{N}_{k,0} + H\mathfrak{N}_{k,1} + H\mathfrak{N}_{k,2} + ... + H\mathfrak{N}_{k,n} \\
&= (\mathfrak{N}_{k,0} + \mathfrak{N}_{k,1}\boldsymbol{j_1} + \mathfrak{N}_{k,2}\boldsymbol{j_2} + \mathfrak{N}_{k,3}\boldsymbol{j_3}) + (\mathfrak{N}_{k,1} + \mathfrak{N}_{k,2}\boldsymbol{j_1} + \mathfrak{N}_{k,3}\boldsymbol{j_2} + \mathfrak{N}_{k,4}\boldsymbol{j_3}) \\
&\quad + ... + (\mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1}\boldsymbol{j_1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_2} + \mathfrak{N}_{k,n+3}\boldsymbol{j_3}) \\
&= (\mathfrak{N}_{k,0} + \mathfrak{N}_{k,1} + \mathfrak{N}_{k,2} + ... + \mathfrak{N}_{k,n}) + (\mathfrak{N}_{k,1} + \mathfrak{N}_{k,2} + \mathfrak{N}_{k,3} + ... + \mathfrak{N}_{k,n+1})\boldsymbol{j_1} \\
&\quad + (\mathfrak{N}_{k,2} + \mathfrak{N}_{k,3} + \mathfrak{N}_{k,4} + ... + \mathfrak{N}_{k,n+2})\boldsymbol{j_2} + (\mathfrak{N}_{k,3} + \mathfrak{N}_{k,4} + \mathfrak{N}_{k,5} + ... + \mathfrak{N}_{k,n+3})\boldsymbol{j_3} \\
&= \sum_{m=0}^{n} \mathfrak{N}_{k,m} + \boldsymbol{j_1} \sum_{m=0}^{n+1} \mathfrak{N}_{k,m} + \boldsymbol{j_2} \left(\sum_{m=0}^{n+2} \mathfrak{N}_{k,m} - 1 \right) + \boldsymbol{j_3} \left(\sum_{m=0}^{n+3} \mathfrak{N}_{k,m} - k - 1 \right) \\
&= \left(\frac{\mathfrak{N}_{k,1} + \mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1} + \mathfrak{N}_{k,n+2} - k\mathfrak{N}_{k,n+1}}{k} \right) \\
&\quad + \left(\frac{\mathfrak{N}_{k,1} + \mathfrak{N}_{k,n+1} + \mathfrak{N}_{k,n+2} + \mathfrak{N}_{k,n+3} - k\mathfrak{N}_{k,n+2}}{k} \right) \boldsymbol{j_1} \\
&\quad + \left(\frac{\mathfrak{N}_{k,1} + \mathfrak{N}_{k,n+2} + \mathfrak{N}_{k,n+3} + \mathfrak{N}_{k,n+4} - k\mathfrak{N}_{k,n+3}}{k} \right) \boldsymbol{j_2} - \boldsymbol{j_2} \\
&\quad + \left(\frac{\mathfrak{N}_{k,1} + \mathfrak{N}_{k,n+3} + \mathfrak{N}_{k,n+4} + \mathfrak{N}_{k,n+5} - k\mathfrak{N}_{k,n+4}}{k} \right) \boldsymbol{j_3} - (k+1)\boldsymbol{j_3} \\
&= \left(\frac{1 + \boldsymbol{j_1} + \boldsymbol{j_2} + \boldsymbol{j_3}}{k} \right) + \left(\frac{\mathfrak{N}_{k,n} + \mathfrak{N}_{k,n+1}\boldsymbol{j_1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_2} + \mathfrak{N}_{k,n+3}\boldsymbol{j_3}}{k} \right) \\
&\quad + \left(\frac{\mathfrak{N}_{k,n+1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_1} + \mathfrak{N}_{k,n+3}\boldsymbol{j_2} + \mathfrak{N}_{k,n+4}\boldsymbol{j_3}}{k} \right) \\
&\quad + \left(\frac{\mathfrak{N}_{k,n+2} + \mathfrak{N}_{k,n+3}\boldsymbol{j_1} + \mathfrak{N}_{k,n+4}\boldsymbol{j_2} + \mathfrak{N}_{k,n+5}\boldsymbol{j_3}}{k} \right) \\
&\quad - k \left(\frac{\mathfrak{N}_{k,n+1} + \mathfrak{N}_{k,n+2}\boldsymbol{j_1} + \mathfrak{N}_{k,n+3}\boldsymbol{j_2} + \mathfrak{N}_{k,n+4}\boldsymbol{j_3}}{k} \right) - \boldsymbol{j_2} - (k+1)\boldsymbol{j_3} \\
&= \frac{H\mathfrak{N}_{k,n} + H\mathfrak{N}_{k,n+1} + H\mathfrak{N}_{k,n+2} - kH\mathfrak{N}_{k,n+1} + 1 + \boldsymbol{j_1} + \boldsymbol{j_2} + \boldsymbol{j_3}}{k} - \boldsymbol{j_2} - (k+1)\boldsymbol{j_3}.
\end{aligned}$$

□

Theorem 2.6. *The generating function for $k-$Narayana hyperbolic quaternions is*

$$\sum_{n=0}^{\infty} H\mathfrak{N}_{k,n}x^n = \frac{H\mathfrak{N}_{k,0} + x(H\mathfrak{N}_{k,1} - kH\mathfrak{N}_{k,0}) + x^2(H\mathfrak{N}_{k,2} - kH\mathfrak{N}_{k,1})}{1 - kx - x^3}. \quad (12)$$

Proof. Suppose that the generating function of the $k-$Narayana hyperbolic quaternions $H\mathfrak{N}_{k,n}$ has the form $f(x) = \sum_{n=0}^{\infty} H\mathfrak{N}_{k,n}x^n$. Then

$$f(x) = H\mathfrak{N}_{k,0} + H\mathfrak{N}_{k,1}x + H\mathfrak{N}_{k,2}x^2 + H\mathfrak{N}_{k,3}x^3 + ... + H\mathfrak{N}_{k,n}x^n + ...$$

Multiply $f(x)$ on both side by kx and then x^3 we have

$$kxf(x) = kH\mathfrak{N}_{k,0}x + kH\mathfrak{N}_{k,1}x^2 + kH\mathfrak{N}_{k,2}x^3 + ... + kH\mathfrak{N}_{k,n-1}x^n + kH\mathfrak{N}_{k,n}x^{n+1} + ...$$
$$x^3 f(x) = H\mathfrak{N}_{k,0}x^3 + H\mathfrak{N}_{k,1}x^4 + H\mathfrak{N}_{k,2}x^5 + ... + H\mathfrak{N}_{k,n-1}x^{n+2} + H\mathfrak{N}_{k,n}x^{n+3} + ...$$

By Lemma 2.2,

$$\begin{aligned}
(1 - kx - x^3)f(x) &= (H\mathfrak{N}_{k,0} + H\mathfrak{N}_{k,1}x + H\mathfrak{N}_{k,2}x^2 + H\mathfrak{N}_{k,3}x^3 + ... + H\mathfrak{N}_{k,n}x^n + ...) \\
&\quad - (kH\mathfrak{N}_{k,0}x + kH\mathfrak{N}_{k,1}x^2 + kH\mathfrak{N}_{k,2}x^3 + ... + kH\mathfrak{N}_{k,n-1}x^n + kH\mathfrak{N}_{k,n}x^{n+1} + ...) \\
&\quad - (H\mathfrak{N}_{k,0}x^3 + H\mathfrak{N}_{k,1}x^4 + H\mathfrak{N}_{k,2}x^5 + ... + H\mathfrak{N}_{k,n-1}x^{n+2} + H\mathfrak{N}_{k,n}x^{n+3} + ...) \\
&= H\mathfrak{N}_{k,0} + x(H\mathfrak{N}_{k,1} - kH\mathfrak{N}_{k,0}) + x^2(H\mathfrak{N}_{k,2} - kH\mathfrak{N}_{k,1}) + x^3(H\mathfrak{N}_{k,3} - (kH\mathfrak{N}_{k,2} + H\mathfrak{N}_{k,0})) \\
&\quad + x^4(H\mathfrak{N}_{k,4} - (kH\mathfrak{N}_{k,3} + H\mathfrak{N}_{k,1})) + ... + x^n(H\mathfrak{N}_{k,n} - (kH\mathfrak{N}_{k,n-1} + H\mathfrak{N}_{k,n-3})) + ... \\
&= H\mathfrak{N}_{k,0} + x(H\mathfrak{N}_{k,1} - kH\mathfrak{N}_{k,0}) + x^2(H\mathfrak{N}_{k,2} - kH\mathfrak{N}_{k,1}) + x^3(H\mathfrak{N}_{k,3} - H\mathfrak{N}_{k,3}) \\
&\quad + x^4(H\mathfrak{N}_{k,4} - H\mathfrak{N}_{k,4}) + ... + x^n(H\mathfrak{N}_{k,n} - H\mathfrak{N}_{k,n}) \\
&= H\mathfrak{N}_{k,0} + x(H\mathfrak{N}_{k,1} - kH\mathfrak{N}_{k,0}) + x^2(H\mathfrak{N}_{k,2} - kH\mathfrak{N}_{k,1}).
\end{aligned}$$

Therefore,

$$f(x) = \frac{H\mathfrak{N}_{k,0} + x(H\mathfrak{N}_{k,1} - kH\mathfrak{N}_{k,0}) + x^2(H\mathfrak{N}_{k,2} - kH\mathfrak{N}_{k,1})}{(1 - kx - x^3)}.$$

□

Theorem 2.7. *For the k−Narayana hyperbolic quaternions, the exponential generating function is*

$$\sum_{n=0}^{\infty} H\mathfrak{N}_{k,n}\frac{x^n}{n!} = \frac{\hat{\varphi}_k\varphi_k e^{\varphi_k x}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\hat{\varpi}_k\varpi_k e^{\varpi_k x}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\hat{\varsigma}_k\varsigma_k e^{\varsigma_k x}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}. \quad (13)$$

Proof. By (10), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} H\mathfrak{N}_{k,n}\frac{x^n}{n!} &= \sum_{n=0}^{\infty}\left(\frac{\varphi_k^{n+1}\hat{\varphi}_k}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\varpi_k^{n+1}\hat{\varpi}_k}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\varsigma_k^{n+1}\hat{\varsigma}_k}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\right)\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{\varphi_k^{n+1}\hat{\varphi}_k x^n}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)n!} + \sum_{n=0}^{\infty}\frac{\varpi_k^{n+1}\hat{\varpi}_k x^n}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)n!} + \sum_{n=0}^{\infty}\frac{\varsigma_k^{n+1}\hat{\varsigma}_k x^n}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)n!} \\
&= \frac{\varphi_k\hat{\varphi}_k}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)}\sum_{n=0}^{\infty}\frac{(\varphi_k x)^n}{n!} + \frac{\varpi_k\hat{\varpi}_k}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)}\sum_{n=0}^{\infty}\frac{(\varpi_k x)^n}{n!} \\
&\quad + \frac{\varsigma_k\hat{\varsigma}_k}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}\sum_{n=0}^{\infty}\frac{(\varsigma_k x)^n}{n!}.
\end{aligned}$$

By the exponential generating function $\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$, we have

$$\sum_{n=0}^{\infty} H\mathfrak{N}_{k,n}\frac{x^n}{n!} = \frac{\hat{\varphi}_k\varphi_k e^{\varphi_k x}}{(\varphi_k - \varpi_k)(\varphi_k - \varsigma_k)} + \frac{\hat{\varpi}_k\varpi_k e^{\varpi_k x}}{(\varpi_k - \varphi_k)(\varpi_k - \varsigma_k)} + \frac{\hat{\varsigma}_k\varsigma_k e^{\varsigma_k x}}{(\varsigma_k - \varphi_k)(\varsigma_k - \varpi_k)}.$$

□

Theorem 2.8. *(Catalan's identity) Given two integers, m and n, such that $m \geq n \geq 0$. Then,*

$$H\mathfrak{R}_{k,n-m}H\mathfrak{R}_{k,n+m} - H\mathfrak{R}_{k,n}^2 = \frac{\hat{\varphi}_k\hat{\varsigma}_k(\varsigma_k-\varphi_k)^{2(m-1)}(\varphi_k\varsigma_k)^{n-m+1}}{(\varphi_k-\varpi_k)(\varpi_k-\varsigma_k)} + \frac{\hat{\varphi}_k\hat{\varpi}_k(\varpi_k-\varphi_k)^{2(m-1)}(\varphi_k\varpi_k)^{n-m+1}}{(\varsigma_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\hat{\varpi}_k\hat{\varsigma}_k(\varsigma_k-\varpi_k)^{2(m-1)}(\varpi_k\varsigma_k)^{n-m+1}}{(\varphi_k-\varpi_k)(\varsigma_k-\varphi_k)}.$$

Proof. By using (10), we have

$$H\mathfrak{R}_{k,n-m}H\mathfrak{R}_{k,n+m} - H\mathfrak{R}_{k,n}^2$$

$$\begin{aligned}
&= \left(\frac{\varphi_k^{n-m+1}\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varpi_k^{n-m+1}\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varsigma_k^{n-m+1}\hat{\varsigma}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\right) \\
&\quad \left(\frac{\varphi_k^{n+m+1}\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varpi_k^{n+m+1}\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varsigma_k^{n+m+1}\hat{\varsigma}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\right) \\
&\quad - \left(\frac{\varphi_k^{n+1}\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varpi_k^{n+1}\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varsigma_k^{n+1}\hat{\varsigma}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\right)^2 \\
&= \frac{\varphi_k^{2n+2}\hat{\varphi}_k^2}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varphi_k^{n-m+1}\varpi_k^{n+m+1}\hat{\varphi}_k\hat{\varpi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} \\
&\quad + \frac{\varphi_k^{n-m+1}\varsigma_k^{n+m+1}\hat{\varphi}_k\hat{\varsigma}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} + \frac{\varpi_k^{n-m+1}\varphi_k^{n+m+1}\hat{\varpi}_k\hat{\varphi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} \\
&\quad + \frac{\varpi_k^{2n+2}\hat{\varpi}_k^2}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varpi_k^{n-m+1}\varsigma_k^{n+m+1}\hat{\varpi}_k\hat{\varsigma}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} \\
&\quad + \frac{\varsigma_k^{n-m+1}\varphi_k^{n+m+1}\hat{\varsigma}_k\hat{\varphi}_k}{((\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varsigma_k^{n-m+1}\varpi_k^{n+m+1}\hat{\varsigma}_k\hat{\varpi}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} \\
&\quad + \frac{\varsigma_k^{2n+2}\hat{\varsigma}_k^2}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} - \frac{\varphi_k^{2n+2}\hat{\varphi}_k^2}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} \\
&\quad - \frac{\varphi_k^{n+1}\varpi_k^{n+1}\hat{\varphi}_k\hat{\varpi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} - \frac{\varphi_k^{n+1}\varsigma_k^{n+1}\hat{\varphi}_k\hat{\varsigma}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} \\
&\quad - \frac{\varpi_k^{n+1}\varphi_k^{n+1}\hat{\varpi}_k\hat{\varphi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} - \frac{\varpi_k^{2n+2}\hat{\varpi}_k^2}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} \\
&\quad - \frac{\varpi_k^{n+1}\varsigma_k^{n+1}\hat{\varpi}_k\hat{\varsigma}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} - \frac{\varsigma_k^{n+1}\varphi_k^{n+1}\hat{\varsigma}_k\hat{\varphi}_k}{((\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} \\
&\quad - \frac{\varsigma_k^{n+1}\varpi_k^{n+1}\hat{\varsigma}_k\hat{\varpi}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} - \frac{\varsigma_k^{2n+2}\hat{\varsigma}_k^2}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} \\
&= \frac{\varphi_k^n\varsigma_k^n\hat{\varphi}_k\hat{\varsigma}_k\left(\frac{(\varsigma_k-\varphi_k)^{2m}}{(\varphi_k\varsigma_k)^{m-1}}\right)}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} + \frac{\varphi_k^n\varpi_k^n\hat{\varphi}_k\hat{\varpi}_k\left(\frac{(\varpi_k-\varphi_k)^{2m}}{(\varphi_k\varpi_k)^{m-1}}\right)}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} \\
&\quad + \frac{\varpi_k^n\varsigma_k^n\hat{\varpi}_k\hat{\varsigma}_k\left(\frac{(\varsigma_k-\varpi_k)^{2m}}{(\varsigma_k\varpi_k)^{m-1}}\right)}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\hat{\varphi}_k\hat{\varsigma}_k(\varsigma_k-\varphi_k)^{2(m-1)}(\varphi_k\varsigma_k)^{n-m+1}}{(\varpi_k-\varphi_k)(\varsigma_k-\varpi_k)} + \frac{\hat{\varphi}_k\hat{\varpi}_k(\varpi_k-\varphi_k)^{2(m-1)}(\varphi_k\varpi_k)^{n-m+1}}{(\varsigma_k-\varphi_k)(\varpi_k-\varsigma_k)}\\
&\quad + \frac{\hat{\varpi}_k\hat{\varsigma}_k(\varsigma_k-\varpi_k)^{2(m-1)}(\varpi_k\varsigma_k)^{n-m+1}}{(\varphi_k-\varpi_k)(\varsigma_k-\varphi_k)}.
\end{aligned}
$$

□

If we set $m = 1$ in Catalan's identity, we have Cassini's identity:

Theorem 2.9. *Assume $n \geq 1$, the following equality is valid:*

$$
H\mathfrak{R}_{k,n-1}H\mathfrak{R}_{k,n+1} - H\mathfrak{R}_{k,n}^2 = \frac{\varphi_k^n\varpi_k^n\hat{\varphi}_k\hat{\varpi}_k}{(\varsigma_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varphi_k^n\varsigma_k^n\hat{\varphi}_k\hat{\varsigma}_k}{(\varphi_k-\varpi_k)(\varpi_k-\varsigma_k)} + \frac{\varpi_k^n\varsigma_k^n\hat{\varpi}_k\hat{\varsigma}_k}{(\varphi_k-\varpi_k)(\varsigma_k-\varphi_k)}.
$$

The following theorem, we present d'Ocagne's identity:

Theorem 2.10. *Given two integers, m and n, such that $m \geq n \geq 0$. Then,*

$$
\begin{aligned}
H\mathfrak{R}_{k,m}H\mathfrak{R}_{k,n+1} - H\mathfrak{R}_{k,m+1}H\mathfrak{R}_{k,n} =& \frac{(\varsigma_k^{n+1}\varphi_k^{n+1})\hat{\varsigma}_k\hat{\varphi}_k(\varsigma_k^{m-n}-\varphi_k^{m-n})}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varphi_k-\varpi_k)} + \frac{(\varphi_k^{n+1}\varpi_k^{n+1})\hat{\varphi}_k\hat{\varpi}_k(\varphi_k^{m-n}-\varpi_k^{m-n})}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varsigma_k)}\\
&+ \frac{(\varpi_k^{n+1}\varsigma_k^{n+1})\hat{\varpi}_k\hat{\varsigma}_k(\varpi_k^{m-n}-\varsigma_k^{m-n})}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)}.
\end{aligned}
$$

Proof. By (10), we obtain
$H\mathfrak{R}_{k,m}H\mathfrak{R}_{k,n+1} - H\mathfrak{R}_{k,m+1}H\mathfrak{R}_{k,n}$

$$
\begin{aligned}
&= \left(\frac{\varphi_k^{m+1}\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varpi_k^{m+1}\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varsigma_k^{m+1}\hat{\varsigma}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\right)\\
&\quad\left(\frac{\varphi_k^{n+2}\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varpi_k^{n+2}\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varsigma_k^{n+2}\hat{\varsigma}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\right)\\
&\quad-\left(\frac{\varphi_k^{m+2}\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varpi_k^{m+2}\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varsigma_k^{m+2}\hat{\varsigma}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\right)\\
&\quad\left(\frac{\varphi_k^{n+1}\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{\varpi_k^{n+1}\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varsigma_k^{n+1}\hat{\varsigma}_k}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\right)\\
&= \frac{\varphi_k^{m+1}\varpi_k^{n+2}\hat{\varphi}_k\hat{\varpi}_k - \varphi_k^{m+2}\varpi_k^{n+1}\hat{\varphi}_k\hat{\varpi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varphi_k^{m+1}\varsigma_k^{n+2}\hat{\varphi}_k\hat{\varsigma}_k - \varphi_k^{m+2}\varsigma_k^{n+1}\hat{\varphi}_k\hat{\varsigma}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\\
&\quad+ \frac{\varpi_k^{m+1}\varphi_k^{n+2}\hat{\varpi}_k\hat{\varphi}_k - \varpi_k^{m+2}\varphi_k^{n+1}\hat{\varpi}_k\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)} + \frac{\varpi_k^{m+1}\varsigma_k^{n+2}\hat{\varphi}_k\hat{\varsigma}_k - \varpi_k^{m+2}\varsigma_k^{n+1}\hat{\varpi}_k\hat{\varsigma}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\\
&\quad+ \frac{\varsigma_k^{m+1}\varphi_k^{n+2}\hat{\varsigma}_k\hat{\varphi}_k - \varsigma_k^{m+2}\varphi_k^{n+1}\hat{\varsigma}_k\hat{\varphi}_k}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)} + \frac{\varsigma_k^{m+1}\varpi_k^{n+2}\hat{\varsigma}_k\hat{\varpi}_k - \varsigma_k^{m+2}\varpi_k^{n+1}\hat{\varsigma}_k\hat{\varpi}_k}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\\
&= \frac{(\varphi_k^2\varsigma_k - \varphi_k\varsigma_k^2)\hat{\varsigma}_k\hat{\varphi}_k(\varsigma_k^m\varphi_k^n - \varphi_k^m\varsigma_k^n)}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)} + \frac{(\varpi_k^2\varphi_k - \varpi_k\varphi_k^2)\hat{\varphi}_k\hat{\varpi}_k(\varphi_k^m\varpi_k^n - \varpi_k^m\varphi_k^n)}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)}\\
&\quad+ \frac{(\varsigma_k^2\varpi_k - \varsigma_k\varpi_k^2)\hat{\varpi}_k\hat{\varsigma}_k(\varpi_k^m\varsigma_k^n - \varsigma_k^m\varpi_k^n)}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)}\\
&= \frac{(\varsigma_k^{n+1}\varphi_k^{n+1})\hat{\varsigma}_k\hat{\varphi}_k(\varsigma_k^{m-n}-\varphi_k^{m-n})}{(\varsigma_k-\varphi_k)(\varsigma_k-\varpi_k)(\varphi_k-\varpi_k)} + \frac{(\varphi_k^{n+1}\varpi_k^{n+1})\hat{\varphi}_k\hat{\varpi}_k(\varphi_k^{m-n}-\varpi_k^{m-n})}{(\varphi_k-\varpi_k)(\varphi_k-\varsigma_k)(\varpi_k-\varsigma_k)}\\
&\quad+ \frac{(\varpi_k^{n+1}\varsigma_k^{n+1})\hat{\varpi}_k\hat{\varsigma}_k(\varpi_k^{m-n}-\varsigma_k^{m-n})}{(\varpi_k-\varphi_k)(\varpi_k-\varsigma_k)(\varsigma_k-\varphi_k)}.
\end{aligned}
$$

□

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