Universal portfolio generated by Hellinger distance

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Abstract. There are numerous universal portfolios generated in different studies using various divergence functions to achieve one goal, which is to maximize wealth. To extend the exploration, through this study, we have generated a new universal portfolio using the Hellinger distance. We conducted a comprehensive performance evaluation of our newly developed universal portfolio against common strategies such as the buy-and-hold strategy and the constant rebalanced portfolio. This evaluation used diverse stock price data from the Bursa Malaysia local platform for trading stocks, bonds, and other securities. Our empirical findings demonstrate that the Hellinger distance-based universal portfolio outperformed the traditional strategies in two out of three datasets over a period of 535 trading days. Specifically, the Hellinger portfolio shows a significant improvement in wealth accumulation, with data set X achieving the highest wealth increase and exhibiting nearly 99% allocation to the most profitable stock. These results highlight the potential of the Hellinger distance-based approach in improving the performance of universal portfolios and offer a promising tool for investment decision-making in today's complex and unpredictable financial markets.

1 Introduction

Investment optimization has always been a critical focus for people, especially in finance, where the aim is to maximize returns. The concept of the universal portfolio, pioneered by Cover [1], significantly impacted this area by suggesting a strategy that could potentially outdo the best stock in the market. Building on this, Helmbold et al. [2] enhanced the universal portfolio concept by employing distance functions in their strategies. This advancement demonstrated the possibility of achieving higher returns than previous methods.

The relevance of this research lies in the rapidly changing nature of financial markets. In a world where markets are unpredictable, strategies that can consistently deliver good returns are highly valued. The universal portfolio strategy, which seeks the best mix of investments for the highest return, is particularly important for investors and portfolio managers.

Our paper aims to extend this line of research. We focus on a novel approach utilizing a specific type of calculation known as the Hellinger distance. This approach has not been widely explored in the context of universal portfolios. Our objective is to test whether integrating this new method can enhance the performance of universal portfolios, thereby offering a more effective tool for investment decision-making.

2 Literature review

The journey of the universal portfolio, a concept that revolutionized investment strategies, began with Thomas M. Cover's seminal work in 1991 [1]. Cover's innovative approach focused on dynamically allocating capital across various assets to optimize cumulative wealth, which both mathematical models and empirical data supported. This foundational idea was then expanded upon in Cover and Ordentlich's study, which introduced the integration of side information into the universal portfolio, aiming to match the wealth of the best side-information-based strategies [3]. Their exploration into different types of universal portfolios, such as uniform and Dirichlet-weighted, further enriched the field.

Building upon these early concepts, Kivinen and Warmuth introduced a framework that became pivotal in developing universal portfolio algorithms [4]. This framework balanced the adjustment of investment strategies against the associated losses, leading to insights into optimization techniques. Following this, Helmbold et al. [2] showcased the effectiveness of online portfolio management using multiplicative update methods. Moving into the early 2000s, Kalai and Vempala focused on the practical aspects of universal portfolios, proposing efficient algorithms based on non-uniform random walks [5]. Their work had broader implications, extending into data compression and language modelling.

In recent years, the scope of research in universal portfolios has broadened considerably. Tan and Kuang experimented with various divergence functions, such as f- divergence and Bregman divergence, to create innovative portfolio strategies in a model-free stock market [6]. Their approach emphasized adjusting investments based on market probabilities. Similarly, He and Yang's study ventured into online portfolio strategies, aggregating expert advice to form a comprehensive approach that aspires to match the performance of the best constant rebalance portfolio [7]. This array of studies illustrates the dynamic and evolving nature of universal portfolio research, continually pushing the boundaries of investment strategy optimization.

In many economic models, transaction costs are often overlooked despite their prevalence in real-world trading conditions. Although this simplification aids in theoretical exploration, it can lead to divergences from reality. Some studies, such as those by Tan and Phoon [8] and Patrick and David [9], have leaned toward these idealistic models. This trend continues in more recent contributions, such as those by Yang and Phoon [10], Ling and Phoon [11], and Bhatt et al. [12], which, while innovative, tend to ignore the practical challenges of transaction costs.

3 Research methodology

The randomness of stock price movement is assumed in a universal portfolio. The main component of Hellinger universal portfolio model namely portfolio vector is defined as $\boldsymbol{b} = (b_1, b_2, \dots, b_i, \dots, b_m)^{\boldsymbol{t}}$ where $\sum_{i=1}^{m} b_i = 1$ and $b_i \geq 0$. It is formed by allocating the investor's wealth in each stock on n^{th} trading day for i^{th} stock, $b_{i,n}$. Another component such as price relative vector is defined as $\boldsymbol{x} = (x_1, x_2, \dots, x_i, \dots x_m)^{\boldsymbol{t}}$, where $x_{i,n} \geq 0$. It is formed by dividing the closing price by the opening price of i^{th} stock on n^{th} trading day, $x_{i,n}$. The wealth of the universal portfolio on $(n+1)^{th}$ trading day is given by:

$$S_{n+1}(\boldsymbol{b}) = \prod_{j=1}^{n+1} \boldsymbol{b_j^t x_j} \tag{3.1}$$

where $S_0(\boldsymbol{b}) = 1$ and $\boldsymbol{b_j^t x_j} = \sum_{i=1} b_{ij} x_{ij}$ [1].

Before introducing our unique approach, it is essential to understand the foundational methods that have shaped the field. The algorithm developed by Helmbold et al. [2], plays a crucial role in this context. Their method involves computing a new portfolio vector, $\boldsymbol{b_{n+1}}$ from the existing portfolio vector $\boldsymbol{b_n}$ and the observed price relatives $\boldsymbol{x_n}$. This process is rooted in the framework by Kivinen and Warmuth [4] for an online prediction based on a linear model. The key aspect of their methodology is the maximization of the following function:

$$F(\boldsymbol{b_{n+1}}) = \xi \log(\boldsymbol{b_{n+1}^t x_{n+1}}) - d(\boldsymbol{b_{n+1}}, \boldsymbol{b_n}), \tag{3.2}$$

where $\xi > 0$ is known as the learning rate and d is a distance measure that serves as a penalty term.

Instead of directly maximize $F(\boldsymbol{b_{n+1}})$, an iterative optimization algorithm, as per Fletcher [13] was suggested in finding the optimal vector $\boldsymbol{b_{n+1}}$. This approach seeks the optimal vector $\boldsymbol{b_{n+1}}$ that fulfils the function $F(\boldsymbol{b_{n+1}})$, under the constraint $\sum_{j=1}^{m} b_{n+1,j} = 1$. However, due to the time-intensive nature of solving different non-linear equations daily, Helmbold et al. suggested a simplification. They proposed using the first-order Taylor series approximation of the first term of Equation (3.2) around $\boldsymbol{b_{n+1}} = \boldsymbol{b_n}$. A Lagrange multiplier is introduced to maintain the sum of the components of $\boldsymbol{b_{n+1}^t}$ equal to 1, leading to the maximization of an adjusted function $\hat{F}$:

$$\hat{F}(\boldsymbol{b_{n+1}}, \lambda) = \xi \left[\log(\boldsymbol{b_n^t x_n}) + \frac{\boldsymbol{b_{n+1}^t x_n}}{\boldsymbol{b_n^t x_n}} - 1\right] - d(\boldsymbol{b_{n+1}}, \boldsymbol{b_n}) + \lambda \left[\sum_{j=1}^{m} b_{n+1,j} - 1\right] \tag{3.3}$$

In our approach, we have adopted the Hellinger distance as the distance function for generating the universal portfolio. The Hellinger distance between two probability distributions $\boldsymbol{p} = (p_i)$ and $\boldsymbol{q} = (q_i)$ is defined by:

$$D_H(\boldsymbol{p}||\boldsymbol{q}) = \sum \left(p^{\frac{1}{2}} - q^{\frac{1}{2}}\right)^2 \tag{3.4}$$

Applying Equation (3.4) to our portfolio vectors, $\boldsymbol{b_n}$ and $\boldsymbol{b_{n+1}}$, the Hellinger distance is represented as:

$$D_H(\boldsymbol{b_{n+1}}||\boldsymbol{b_n}) = \sum_{i=1}^{m} \left(b_{n+1,i}^{\frac{1}{2}} - b_{n,i}^{\frac{1}{2}}\right)^2 \tag{3.5}$$

Equation (3.5) replaces the distance measure in our objective function, leading to a new formulation:

$$\hat{F}(\boldsymbol{b_{n+1}},\lambda) = \xi\left[\log(\boldsymbol{b_{n+1}^t x_{n+1}}) + \frac{\boldsymbol{b_{n+1}^t x_n}}{\boldsymbol{b_n^t x_n}} - 1\right] - \sum_{i=1}^{m}\left(b_{n+1,i}^{\frac{1}{2}} - b_{n,i}^{\frac{1}{2}}\right)^2 + \lambda\left[\sum\nolimits_{j=1}^{m} b_{n+1,j} - 1\right] \tag{3.6}$$

To determine the optimal portfolio vector $\boldsymbol{b_{n+1}}$, we derive the Hellinger universal portfolio by setting the derivative of $\hat{F}$ with respect to $b_{n+1,i}$ and set it equal to 0. This results in a formula that updates each component of the portfolio vector based on the learning rate ξ, the price relative $\boldsymbol{x_n}$, and a Lagrange multiplier λ, ensuring that the sum of the components equals 1.

The final expression for updating the portfolio vector is:

$$b_{n+1,i} = \frac{b_{n,i}\left[\beta(\boldsymbol{b_n^t x_n}) - x_{n,i}\right]^{-2}}{\sum_{j=1}^{m} b_{n,j}\left[\beta(\boldsymbol{b_n^t x_n}) - x_{n,i}\right]^{-2}} \tag{3.7}$$

where β is a real parameter.

Proof:

Differentiate Equation (3.6) with respect to $b_{n+1,i}$, it gives:

$$\frac{\partial\hat{F}}{\partial b_{n+1,i}} = \xi\left[\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}}\right] - \left(1 - b_{n+1,i}^{-\frac{1}{2}} b_{n,i}^{\frac{1}{2}}\right) + \lambda \tag{3.8}$$

for $i = 1,2,\dots,m$. Setting Equation (3.8) equals 0 and restructure the equation, leading to:

$$\xi\left[\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}}\right] - \left(1 - \left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}}\right) + \lambda = 0 \tag{3.9}$$

for $i = 1,2,\dots,m$. Multiply $b_{n,i}$ with Equation (3.9) and sum over i to obtain:

$$\lambda = \left[1 - \sum_{i=1}^{m} b_{n,i}\left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}}\right] - \xi \tag{3.10}$$

The variable λ obtained in Equation (3.10) is the Lagrange multiplier. Substitute the Equation (3.10) into Equation (3.9) to get:

$$\xi\left[\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}}\right] - \left(1 - \left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}}\right) + \left[1 - \sum_{i=1}^{m} b_{n,i}\left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}}\right] - \xi = 0 \tag{3.11}$$

Rearrange the Equation (3.11) into:

$$\xi\left[\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}} - 1\right] + \left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}} - \sum_{i=1}^{m} b_{n,i}\left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}} = 0 \tag{3.12}$$

for $i = 1,2,\dots,m$.

Let

$$Z_i = b_{n,i}\left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}} \tag{3.13}$$

Substitute Equation (3.13) into the Equation (3.12), and leading to:

$$\xi\left[1-\left(\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}}\right)\right]-\left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}}+\sum_{i=1}^{m} Z_i = 0 \tag{3.14}$$

for $i = 1,2,\dots,m$.

Let

$$\eta = \sum_{i=1}^{m} Z_i \tag{3.15}$$

for $i = 1,2,\dots,m$, where η does not depend on i. Then Equation (3.14) becomes

$$\xi\left[1-\left(\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}}\right)\right]-\left(\frac{b_{n,i}}{b_{n+1,i}}\right)^{\frac{1}{2}}+\eta = 0 \tag{3.16}$$

Rearrange the Equation (3.16), and leading to:

$$\begin{aligned} b_{n+1,i} &= \frac{b_{n,i}}{\left\{\eta+\xi\left[1-\left(\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}}\right)\right]\right\}^2} \\ &= \frac{b_{n,i}}{\left[\xi\left(\frac{\eta}{\xi}+1-\frac{x_{n,i}}{\boldsymbol{b_n^t x_n}}\right)\right]^2} \\ &= \frac{b_{n,i}}{\left(\frac{\xi}{\boldsymbol{b_n^t x_n}}\right)^2\left[\left(\frac{\eta}{\xi}+1\right)(\boldsymbol{b_n^t x_n})-x_{n,i}\right]^2} \end{aligned} \tag{3.17}$$

for $i = 1,2,\dots,m$.

Let

$$\beta = \left(\frac{\eta}{\xi}+1\right) \tag{3.18}$$

where β is a real parameter. Therefore Equation (3.17) becomes:

$$b_{n+1,i} = \left(\frac{\boldsymbol{b_n^t x_n}}{\xi}\right)^2\left[b_{n,i}\left(\beta(\boldsymbol{b_n^t x_n})-x_{n,i}\right)^{-2}\right] \tag{3.19}$$

for $i = 1,2,\dots,m$. We sum Equation (3.19) over i to obtain:

$$\left(\frac{\boldsymbol{b}_{\boldsymbol{n}}^{\boldsymbol{t}}\boldsymbol{x}_{\boldsymbol{n}}}{\xi}\right)^2 \sum\nolimits_{i=1}^{m} \left[b_{n,i}\left(\beta(\boldsymbol{b}_{\boldsymbol{n}}^{\boldsymbol{t}}\boldsymbol{x}_{\boldsymbol{n}}) - x_{n,i}\right)^{-2}\right] = 1 \tag{3.20}$$

Rearrange the equation (3.20), and leading to:

$$\left(\frac{\boldsymbol{b}_{\boldsymbol{n}}^{\boldsymbol{t}}\boldsymbol{x}_{\boldsymbol{n}}}{\xi}\right)^2 = \frac{1}{\sum_{i=1}^{m}\left[b_{n,i}\left(\beta(\boldsymbol{b}_{\boldsymbol{n}}^{\boldsymbol{t}}\boldsymbol{x}_{\boldsymbol{n}}) - x_{n,i}\right)^{-2}\right]} \tag{3.21}$$

and finally, we combine Equation (3.19) and Equation (3.21), leading to:

$$b_{n+1,i} = \frac{b_{n,i}\left(\beta(\boldsymbol{b}_{\boldsymbol{n}}^{\boldsymbol{t}}\boldsymbol{x}_{\boldsymbol{n}}) - x_{n,i}\right)^{-2}}{\sum_{i=1}^{m}\left[b_{n,i}\left(\beta(\boldsymbol{b}_{\boldsymbol{n}}^{\boldsymbol{t}}\boldsymbol{x}_{\boldsymbol{n}}) - x_{n,i}\right)^{-2}\right]}$$

Equation (3.7) is obtained.

4 Empirical performance

The selected stock data sets encompass trading data from 26th October 2020 to 30th December 2022, covering a total of 535 trading days. This duration provides a comprehensive view of market behaviour and stock performance over an extended period, offering a robust basis for our analysis. The composition of each portfolio is designed to reflect a cross-section of the Malaysian market, ensuring a broad representation of market trends and patterns.

Table 1. Stock Dataset for Empirical Study.

Dataset	Portfolio Composition: Malaysian Companies Listed in Bursa Malaysia
X	CIMB Group Holdings Berhad, Mr. D.I.Y Group (M) Berhad, British American Tobacco (M) Berhad
Y	Hap Seng Consolidated Berhad, Maxis Berhad, Nestle (M) Berhad
Z	Astro Malaysia Holding Berhad, IJM Corporation Berhad, IHH Health Berhad

Table 1 shows an overview of the stocks included in each dataset. It outlines the specific companies whose stocks from the basis of datasets X, Y, and Z. Each dataset is crafted to provide a unique combination of stocks, reflecting different market segments and investment profiles. The diversity within and across these datasets is integral to our analysis, enabling a comprehensive evaluation of the Hellinger universal portfolio strategy against a backdrop of real-market conditions and diverse investment scenarios.

In this study, we analyzed the Hellinger universal portfolio's performance using three datasets from Bursa Malaysia, each comprising stocks from three different companies. We began with an equal-weight initial portfolio vector $b_1 = (0.3333{,}0.3333{,}0.3334)$ for all datasets. The parameter β was varied from 9.1 to 10.0 to optimize the portfolio's performance. The terminal wealth S_{535} and the concluding portfolio vectors $\boldsymbol{b}_{\boldsymbol{536}}$ were evaluated for the most optimal β value, as detailed in Tables 2, 3 and 4. The empirical results demonstrated notable differences in wealth accumulation across datasets X, Y, and Z. Dataset X showed the highest wealth increase at 2.2633 units, achieved with $\beta = 10.0$, where nearly 99% of the portfolio was allocated to the first company's stock. In contrast, dataset Z generated the least wealth, but with a significant allocation (99.9%) to the third stock, indicating its strong performance.

Table 2. The accumulated wealth by running Hellinger universal portfolio after 535 trading days over dataset X for selected parameter β with the next day portfolio vector $\boldsymbol{b_{536}}$ where the initial portfolio, $b_1 = (0.3333{,}0.3333{,}0.3334)$.

Set	β	S_{535}	b_1	b_2	b_3
X	9.1	2.2612	0.9933	0.0066	0.0001
	9.2	2.2619	0.9938	0.0062	0.0000
	9.3	2.2621	0.9942	0.0058	0.0000
	9.4	2.2623	0.9945	0.0055	0.0000
	9.5	2.2625	0.9948	0.0051	0.0000
	9.6	2.2627	0.9952	0.0048	0.0000
	9.7	2.2628	0.9954	0.0045	0.0000
	9.8	2.2630	0.9957	0.0043	0.0000
	9.9	2.2632	0.9960	0.0040	0.0000
	10.0	2.2633	0.9962	0.0038	0.0000

Table 3. The accumulated wealth by running Hellinger universal portfolio after 535 trading days over dataset Y for selected parameter β with the next day portfolio vector $\boldsymbol{b_{536}}$ where the initial portfolio, $b_1 = (0.3333{,}0.3333{,}0.3334)$.

Set	β	S_{535}	b_1	b_2	b_3
Y	9.1	1.4305	0.1153	0.0016	0.8836
	9.2	1.4307	0.1129	0.0014	0.8861
	9.3	1.4309	0.1106	0.0013	0.8885
	9.4	1.4311	0.1084	0.0012	0.8908
	9.5	1.4313	0.1063	0.0011	0.8930
	9.6	1.4315	0.1043	0.0010	0.8952
	9.7	1.4317	0.1023	0.0009	0.8972
	9.8	1.4319	0.1003	0.0008	0.8992
	9.9	1.4320	0.0985	0.0007	0.9011
	10.0	1.4322	0.0967	0.0007	0.9030

Table 4. The accumulated wealth by running Hellinger universal portfolio after 535 trading days over dataset Z for selected parameter β with the next day portfolio vector $\boldsymbol{b_{536}}$ where the initial portfolio, $b_1 = (0.3333{,}0.3333{,}0.3334)$.

Set	β	S_{535}	b_1	b_2	b_3
Z	9.1	1.1164	0.0000	0.0002	0.9998
	9.2	1.1165	0.0000	0.0001	0.9999
	9.3	1.1165	0.0000	0.0001	0.9999
	9.4	1.1165	0.0000	0.0001	0.9999
	9.5	1.1165	0.0000	0.0001	0.9999
	9.6	1.1165	0.0000	0.0001	0.9999
	9.7	1.1165	0.0000	0.0001	0.9999
	9.8	1.1165	0.0000	0.0001	0.9999
	9.9	1.1165	0.0000	0.0001	0.9999
	10.0	1.1165	0.0000	0.0001	0.9999

We further compared the Hellinger universal portfolio with the traditional buy-and-hold strategy. Tables 5, 6, and 7 compare the performances of buy-and-hold strategy and Hellinger universal portfolio in Datasets X, Y, and Z, respectively. For Dataset X, the Hellinger strategy yielded a 127% wealth increment, outperforming the buy-and-hold strategy's increments of 90.2%, 72.4%, and 12.5% for individual stocks. In Table 6, the performance analysis between the traditional buy-and-hold strategy and the Hellinger universal portfolio strategy for Dataset Y reveals a striking contrast. The buy-and-hold strategy resulted in substantial wealth depreciation, with stock values falling between 11.7% and 22.4%, signaling a downturn in individual stock performance over the assessed period. Contrary to this, the Hellinger universal portfolio strategy successfully navigated the bearish trends, achieving a significant wealth increment of 42%. This impressive performance is attributed to the portfolio's strategic reallocation of assets, which adeptly mitigated losses and capitalized on any available gains, thereby demonstrating the robustness and adaptive efficiency of the Hellinger method in optimizing investment returns despite challenging market conditions. Dataset Z's analysis showed a more nuanced picture, with the Hellinger portfolio reporting a modest 12% wealth increase, while the buy-and-hold strategy presented a mixed performance, surpassing the Hellinger approach in two out of three stocks.

These observations suggest that the Hellinger universal portfolio, especially in turbulent market conditions like those during the COVID-19 pandemic, can effectively reallocate resources to maximize wealth. The healthcare sector, as represented in Dataset Z, showed notable performance, likely influenced by the pandemic's impact on the market.

Table 5. Comparison table of wealth increment for Buy-and-hold strategy and Hellinger universal portfolio for Dataset X.

Stock	**First Day Opening Price**	**545th Day Closing Price**	**Buy-and-Hold wealth increment**	**Hellinger Wealth increment**	**Portfolio vector, $\boldsymbol{b_{535}}, \boldsymbol{\beta = 10}$**
1	3.05	5.8	90.2%	127%	0.9962
2	1.16	2.00	72.4%		0.0038
3	9.97	11.22	12.5%		0.0000

Table 6. Comparison table of wealth increment for Buy-and-hold strategy and Hellinger universal portfolio for Dataset Y.

Stock	**First Day Opening Price**	**545th Day Closing Price**	**Buy-and-Hold wealth increment**	**Hellinger Wealth increment**	**Portfolio vector, $\boldsymbol{b_{535}}, \boldsymbol{\beta = 10}$**
1	7.25	6.40	-11.7%	42%	0.0967
2	4.95	3.84	-22.4%		0.0007
3	140.89	140.00	-0.6%		0.9030

Table 7. Comparison table of wealth increment for Buy-and-hold strategy and Hellinger universal portfolio for Dataset Z.

Stock	**First Day Opening Price**	**545th Day Closing Price**	**Buy-and-Hold wealth increment**	**Hellinger Wealth increment**	**Portfolio vector, $\boldsymbol{b_{535}}, \boldsymbol{\beta = 10}$**
1	0.74	0.65	11.6%	12%	0.0000
2	1.38	1.60	15.9%		0.0000
3	5.10	6.22	22.0%		0.9999

5 Conclusion

This research ventured to construct and evaluate a new universal portfolio, harnessing the Hellinger distance function for its formulation and gauging its efficacy with historical stock data. Our findings indicate a commendable performance of the Hellinger-inspired portfolio, which surpassed the conventional buy-and-hold strategy in two of the three datasets studied over 535 trading days, underscoring the significant promise of the Hellinger method in bolstering investment returns. However, a notable underperformance in dataset Z calls attention to the limitations of the universal portfolio algorithm, particularly in markets or stock scenarios that deviate from the norm, thereby stressing the algorithm's sensitivity to varying market dynamics.

While the Hellinger universal portfolio has shown a 67% success rate in outperforming benchmark strategies and holds substantial potential for optimizing investment strategies, our conclusions must prudently acknowledge the need for further empirical scrutiny. It is imperative that future research encompasses a broader spectrum of comparative analyses with diverse methodologies, especially those that have demonstrated consistent results across varying market conditions. Such comparative studies will be pivotal in validating the robustness of the Hellinger approach. Additionally, integrating considerations such as market condition analysis and transaction costs into the universal portfolio framework could significantly refine and tailor the strategy, enhancing its versatility and application across a multifaceted investment landscape (Blum and Kalai [14], Kor et al. [15]). Hence, we advocate for continued and expanded research to ascertain the most reliable and effective strategies in the domain of investment management.

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