Visual mathematics: an implementation to students in an indigenous community and a sub urban area

Jalina Widjaja[1*], *Oki* Neswan[1], and *Yudi* Soeharyadi[1]

[1]Departemen of Mathematics, Institut Teknologi Bandung, Indonesia

Abstract. Visual mathematics is a mathematical exploration involving the senses and provides students with models and images for concepts, enhancing mathematics learning. Hands-on activities are central to educe and nurture students' logic, observational skills, and habits. Visual mathematics may also be useful for communicating mathematical concepts that minimizes the dependence on culture and language. We propose two modules on visual mathematics for year 4 elementary school students, the first in the arithmetic operation of fractions, and the second in basic planar figures, their perimeters, and areas. The construction is not only appealing figuratively but more importantly, it maintains precise mathematical constructions.

1 Introduction

Mathematics, at its core, is a visual and spatial discipline. The fundamental concepts and relationships in mathematics are often best understood through visual representations and intuitions (see for example Boaler [5], Presmeg[15]). In accordance to Zimmermann and Cunningham (1991) [25], as well as Hershkowitz et al. (1989) [10], Arcavi (2003)[2] defines visualization as follows: "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings". Furthermore, one of the strong justifications for visual mathematics can be grounded to modern brain theory. According to some results in neuroscience research about the ways human brains work when they study and learn mathematics, the brain activity is spread out across a widely-distributed network, which include two visual pathways: the ventral and dorsal visual pathways. Neuroimaging has shown that even when people work on a number calculation, such as 12×25, with symbolic digits (12 and 25) our mathematical thinking is grounded in visual processing [6].

From the geometric proofs of ancient Greece to the complex visualizations used in modern mathematical research, the ability to effectively communicate and reason about mathematical ideas through visual means has been a crucial aspect of the field. Diagrams,

* Corresponding author: yalina62@gmail.org

charts, and other math-based visualizations can convey complex quantitative information in a way that transcends the need for written or spoken translation. A graph illustrating economic trends, for example, can be interpreted and understood by people from vastly different cultural backgrounds.

Although ideally mathematical ideas are universal, and therefore they transcend cultures and languages, in real world, especially in teaching and learning mathematics, linguistic and cultural differences can often be barriers for students to communicate and understand. To overcome these barriers, visual mathematics is proposed. The teaching and learning of mathematics have increasingly incorporated more visual elements in recent decades. From number lines to geometric proofs, the visual components of math make the subject more accessible and intuitive for students. This visual approach has been especially beneficial for students who may be learning math in a language that is not their native tongue. The diagrams and symbolic representations allow them to grasp the concepts without being hindered by the vocabulary.

For young learners, visual mathematics is a vehicle for them to learn mathematics, and the world around them as well. It should be an integral and strategic part of learning mathematics in schools. Visual mathematics is an integral part of the proposal for a STEAM education framework for schools in indigenous communities [20]), exactly for the reason of its capacity to transcend potential language and cultural barriers.

Proficiency in visual representation is essential for success in advanced mathematical and scientific courses, as these subjects heavily rely on visualization and spatial reasoning to solve complex problems (Zhang et al [22]). Thus, it is not surprising that many authors emphasize the significance of visual thinking in the learning process and provide compelling arguments regarding the pivotal role of visualization in the reform of mathematics teaching [23,2,13]. However, they also acknowledge the potential challenges and constraints associated with visualization, as indicated by other studies [14,3,4].

Despite the significance of visualization in mathematics, research by Presmeg & Bergsten [15] and Healy & Hoyles [7] reveals that students generally exhibit reluctance to engage in visualization in mathematics. Hence it raises an intriguing question: how can students best be encouraged to employ this way of thinking?

This article is a preliminary report of our ongoing project to construct visual mathematics modules for elementary school students. The objective of the project is to write and construct visual mathematics modules to serve two purposes, those are:

1. As an accompanying module in traditional math teaching, so the teacher can deliver the topics in a proper logical order. Also, the students can be motivated to learn the topics due to the figures used in the module. For this purpose, the teacher should carefully pick the figures to be discussed, to meet the time limit and also to enable the students to learn independently.
2. As an independent learning material that can be used by the students to learn the topics independently, regardless of the language used in the module. Hence the concepts in this module are slowly but rigorously built using a series of figures that are quite common or familiar to most of the students. For this purpose, the students should be conditioned to use the module. Keen observation is one of the key points required for learning the topics using the module. Some games involving observations can be given in the beginning.

In designing our modules, the major steps are:

1. Construct an algorithm to introduce mathematical concepts using pictures. The pictures should not be only illustrative, but furthermore it should carry the mathematical idea correctly, and also suitable to the cognitive development of students,

2. The objects in the pictures can be chosen to be universal or common objects in most of all communities, such as trees, fruits, leaves, stones. The pictures can be made more interesting by taking more relevant objects to the students, or the communities, and thus, the modules designed using visual mathematics can be adopted with minimum effort to be used in various communities.

The first module which covers the arithmetic of fractions was trialed in the elementary school in the indigenous community of Kasepuhan Ciptagelar, in South Sukabumi, in August 2023. The second module covers the geometry of simple planar shapes and was trialed in elementary schools in Lembang, West Bandung Regency in September 2024. The first location was chosen to represent students in remote and isolated communities. Kasepuhan Ciptagelar is an indigenous community based on its sustainable rice-paddy agriculture. One of the unique features of the Kasepuhan Ciptagelar is their openness to technology (including somewhat regulated internet access), despite their strict adherence to their culture. Their school facilities and infrastructures are still minimal, such as the poor ratio of students-teachers. However, their limited bandwidth of internet and low ownership of devices, still enables them to limited exposure to various learning material. The second location was selected to represent students in schools in sub urban areas. Lembang is a suburb of Bandung City. Despite its vicinity to large city, Lembang still maintains the mix characteristics of a city and kampong, in terms of socio-economic-cultural values and conditions, access to modernity, and etc. These two choices represent somewhat contrasting background conditions for the implementation of visual mathematics. The second module was created with feedback from the first trial.

2 The modules

In making visualizations of concepts in mathematics, a detailed process of building or describing the concepts should be done. It is needed to make the students learn the concepts rigorously, comprehensively, and effectively.

In the beginning of the process, it is important to make the students familiar with this method is by guiding them directly in the class or using some preliminary activities. The students are asked to look closely at the series of figures given in the module and describe the process that is represented by those figures.

When a concept is introduced, we have to find the basic ideas so that the advanced topic can be built based on these basic ideas while retaining consistency. The visual objects used in the modules were tailored to students' backgrounds and environments. Exercise problems that follow, inquire students to show their understanding by sketching series of pictures related to the concepts. This type of exercise not only shows the students' understanding, but also let them express their creativity. The freedom in expressing their understanding may help the students develop the joy of learning mathematics.

2.1 Fractions and their arithmetic

Fractions are fundamental concepts in mathematics, used to represent the concept of parts of a whole. Since fractions form the basis for more advanced mathematical concepts, mastery of fractions and their arithmetical operations is crucial for students [1]. It is crucial to their capacity to successfully grasp new ideas like calculus, algebra, probability, and trigonometry [5]. Even though fractions are taught starting in elementary school, many adults, including high school students, continue to struggle with fractions misconceptions [19,21,16]. Consequently, it is imperative to find an alternative approach.

2.1.1 Arithmetical Operations of Fraction

Elementary school students often find arithmetic operations, such as addition, subtraction, multiplication, and division of fractions, to be challenging concepts to grasp [10,9] Hence, we consider it very critical to find an alternative method to explain these operations, which helps students build conceptual understanding [1].

There are several methods of approaching the problem and some have proved to be successful [9]. There is still the need to minimize the difficulty of understanding the concept. By employing a visual approach to introduce mathematical concepts, it is expected that students can avoid the misconceptions that sometimes arise. However, studies have also shown that young students can successfully divide a given number of items among persons and fail to do so when the same problem is presented simply with symbolic notation and no visual clues [17].

Therefore, employing visual representation is highly likely to help students build a solid understanding of the concept.

2.1.2 Fractions

Some researchers claimed that the process of dividing a unit into equal parts is crucial for developing a comprehensive understanding of rational numbers [6,11,18]. Thus, the idea behind our visualizations is based on the notion that a fraction represents an object or collection of objects that are equal in part. For instance, a pizza or a pie is cut into four equal slices or parts. Each slice represents $\frac{1}{4}$ of the whole. Thus, the fraction $\frac{3}{4}$ represents a collection of three of them.

Fractions can have the same value; in that case, we refer to them as equal or equivalent. We need this notion when we compare two fractions or to add two fractions that have different denominators.

2.1.3 Least Common Multiples

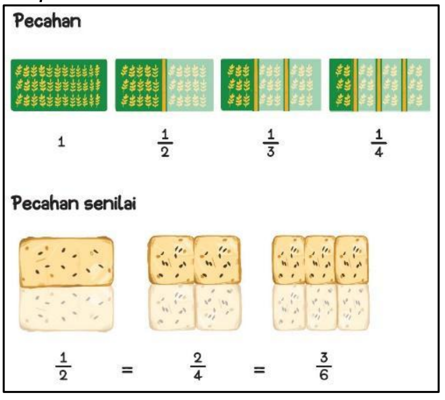

Fig. 1. Equivalent fractions

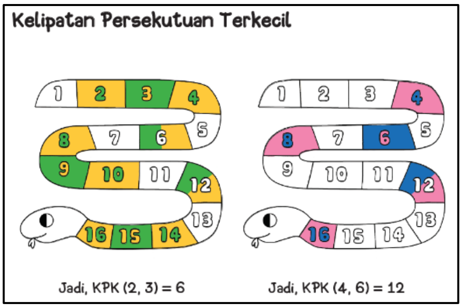

Fig. 2. Least Common Multiple

In order to determine the common denominator, students must possess an understanding of common multiples, but the least common multiple (lcm) is the most efficient method because usually it avoids the need to simplify the result. The concept of least common multiples can be visualized as seen in the following.

2.1.4 Addition

When fractions have the same denominator, the addition operation is in general straightforward. Every fraction addition problem is known to be reducible to a fraction addition problem with the same denominator.

1. Check if the denominators are the same. If not, reduce the problem into a problem of fractions addition with the same denominator, say $\frac{n}{m} + \frac{p}{m}$.
2. Cut each two pies (or rectangles) into m equal parts. For each pie, shade the appropriate number of slices to reflect the fractions.
3. The total number of shaded slices is the numerator of the fraction that results from the addition process. The denominator is m.

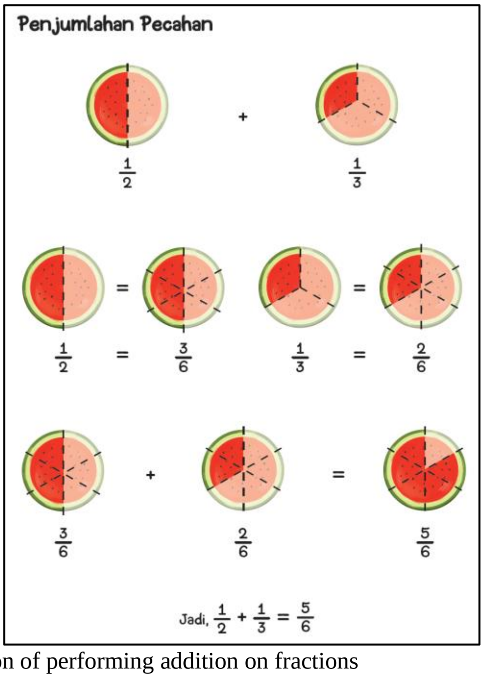

Fig. 3. Visual representation of performing addition on fractions

2.1.5 Multiplication

Compared to addition, the concept of fractions multiplication is more challenging, even though the operation is more straightforward. Making the concept relatable to students is important to help students understand it. Multiplication of fractions focuses on finding a portion of a portion. Multiplication of fractions asks, "What part of the whole do you get when you take a part of a part?" Therefore, it is very sensible to start with the most straight forward case, which is multiplications involving unit fractions. We can find the multiplication $\frac{n}{m} \times \frac{p}{k}$ as follows.

1. Cut a rectangle vertically into m equal parts and then shade the n slices to represent the $\frac{n}{m}$ fraction.
2. Cut each slice of the rectangle into k parts equally. Thus, each thin slice corresponds to $\frac{1}{mk}$ of the rectangle.
3. For each slice, shade p thin slices.
4. Then the number of shaded thin slices is the numerator of the fraction that arises from the multiplication process. In the meantime, mk is the fraction's denominator.
5.

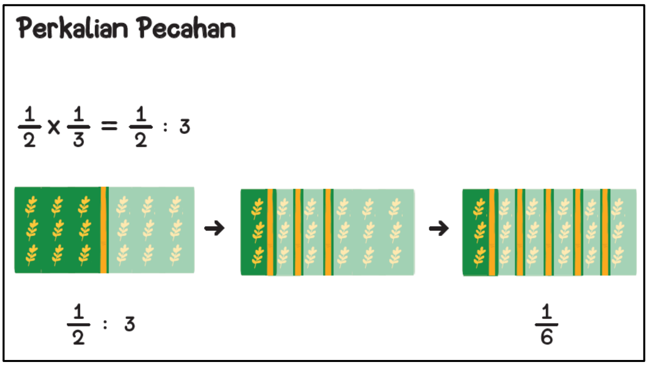

Fig. 4. Unit fractions multiplication

2.1.6 Division

The division is essentially the inverse of multiplication. To put in words, division asks "How many of these fit into that?" Consequently, it is best to go over the division's verbal meaning to the students before moving into the division operation.

We may ask students to relate the division problem $\frac{1}{2}:\frac{1}{4}$ to the question "How many quarters of bread fit into a half of bread?" using this verbal interpretation. The visual explanation in Figure 4 should help students to find that the answer is 2.

However, the answers to division problems are not necessarily integers. Nevertheless, the same meaning still holds in this case. A question like "How much of this fits into that?" can be used to verbalize the division $\frac{f_1}{f_2}$, if f_1 and f_2 are fractions such that $f_1 < f_2$. For instance, $\frac{1}{6}:\frac{1}{3} = \frac{1}{2}$ indicates that half of a third of a pizza fits into a sixth of a pizza. It would be easier to answer this question if the fractions were expressed using the same denominator. The expression of the fractions using the same denominator enables us to visualize that a sixth pizza is half of a third pizza.

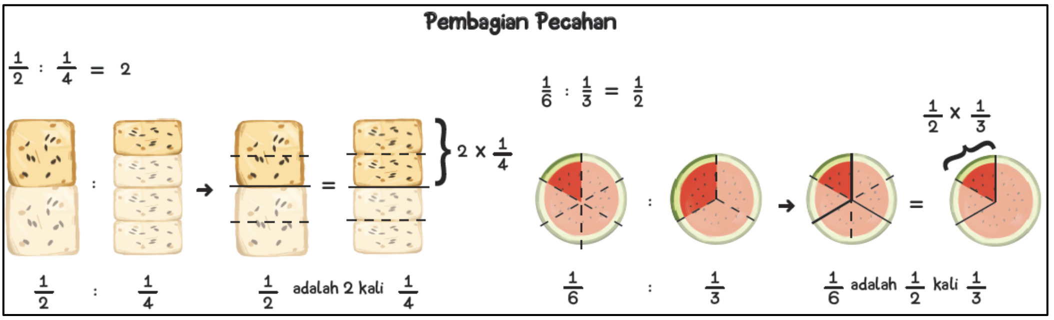

Fig. 5. Division of fractions

2.1.7 Ordering Fraction

Sometimes we need to find out which of two fractions is greater than the other one. Therefore, we need to know how to order two fractions. When ordering fractions, the most important thing to remember is that the denominators must be the same, thus the order is determined by the numerators. The same method is used in some arithmetic operations on fractions, those are addition, subtraction, and division.

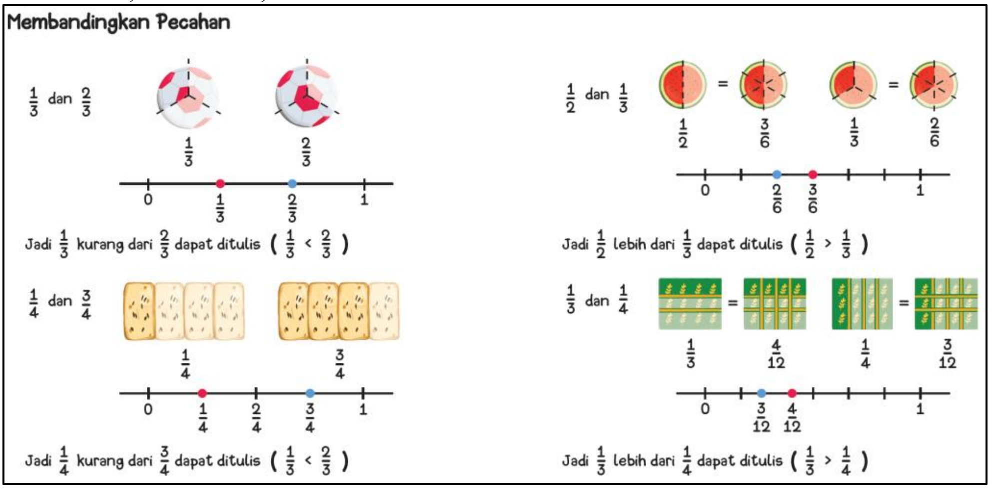

Fig. 6. Ordering fractions.

2.2 Shape and their measurements

According to the National Council of Teachers of Mathematics (NCTM), a key objective of learning geometry is to develop students' visual awareness and make them visually literate persons. NCTM states that geometry will lead students to analyze the properties of geometric shapes by using spatial thinking and geometrical modeling in solving problems. Therefore, it was emphasized that learning geometry will help students to become more proficient in reasoning and justification [12].

In the twenty-first century, the capacity to analyze digital, visual, and audio information is as fundamental as reading and writing. In this regard, geometry instruction should help students maximize the benefits of the visual materials among the course materials [8].

Thus, the topic discussed in the second module is basic planar shapes, especially their perimeter and area. We use the main ideas as follows.

1. The perimeter of a planar shape is the sum of the length of its sides. It includes introducing the Pythagorean Theorem.
2. The concept of the area of a planar shape is introduced using a decomposition of the shape into right triangles. The area of a planar shape is the sum of the areas of the right triangles inscribed in the shape.

The detailed process of introducing the concepts of perimeter and area of planar shapes is as follows.

1. A line segment is introduced with its length. Some real objects in the shape of line segments are given, with their length.

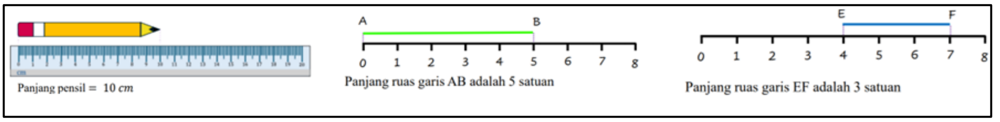

Fig. 7. Length of segments

2. A square is introduced whose sides are line segments. The perimeter of the square is discussed using the length of its sides. The area of a square whose length of each side is 1 unit is defined as one unit square. Then the area of a square is defined according to the number of the unit square inscribed in the square.

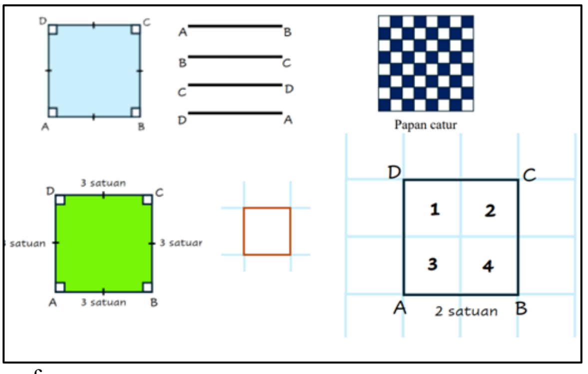

Fig. 8. Perimeter and area of squares

3. The perimeter and area of a rectangle are introduced as the ones for the square.
4. A diagonal of a rectangle is introduced, as well as its length using the Pythagorean Theorem.

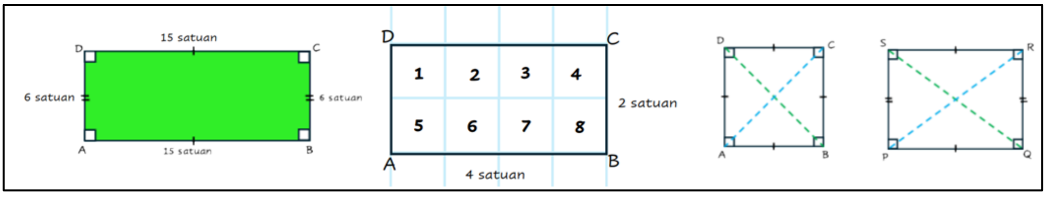

Fig. 9. Perimeter and area of rectangles

6. A right triangle can be obtained by dividing a rectangle into two equal parts along one of its diagonals.

7. The perimeter of a right triangle is introduced as the sum of the length of its sides. The area of a right triangle is defined as half of the area of the corresponding rectangle.

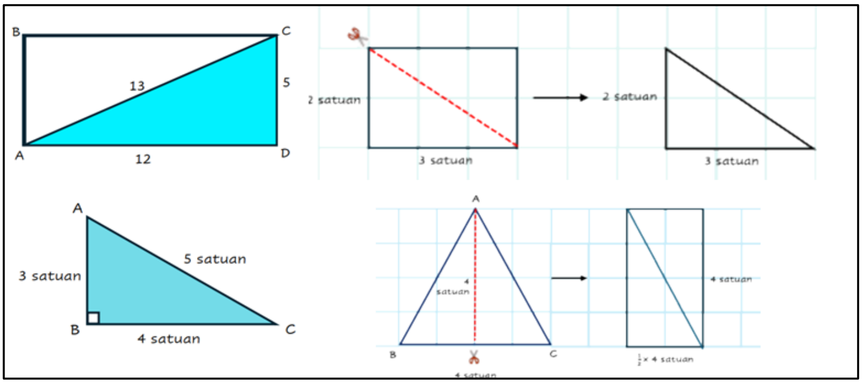

Fig. 10. Perimeter and area of triangles

8. The perimeter of a parallelepiped, trapezoid, kite, and rhombus are introduced as the sum of the length of their sides.
9. The area of a parallelepiped, trapezoid, kite, and rhombus are introduced as the sum of the area of the right triangles inscribed in each shape.

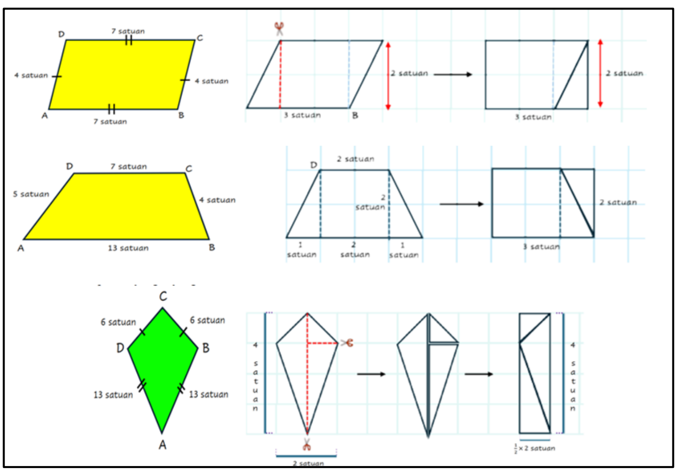

Fig. 11. Perimeter and area of quadrilaterals

10. A series of figures are given to guide the students in finding the formula of the perimeter and area of each shape.
11. Some exercise problems, with increasing levels of complexity, are placed following the given examples.

The figures in the module as examples or in exercise problems are varied. Usually, they begin with some real objects that are familiar to the students. Then gradually they become more abstract as the concepts are introduced for general objects with a certain shape.

3 Conclusion and Further plan

Visual mathematics is proposed to be a method of learning mathematics that can be used directly, or adapted quite easily if required, to various communities with their own cultures and languages. Two modules that are constructed using this method have been trialed to two groups of students. One group of students is in an indigenous community, the other is in the sub urban area.

The modules can be used as a main reference for teachers in the class when they deliver the topic. The modules also help to guide the teacher and students in discussing the topic due to their detailed steps of the process building the concepts in a precise, detailed, rigorous manners. The teachers can help the students to learn more effectively, and the students are quite motivated because of the figures used in the modules. The modules can also be used by the students as an independent study guide, with more time allowance. It requires keen, logical, and critical observations. Some preconditioning exercises for the students are strongly suggested.

Some challenges in making the modules are picking the strategy to deliver the concepts, how detailed the process of building the concepts is shown, and the objects used to describe the process. The strategy is selected based on the compatibility of using figures in explaining the concepts. The flow of presenting the concepts relates to the ability of the students to learn. The objects used in the figure should be familiar and quite common to the students.

A quatitative study to measure the effectiveness of the method is planned as a follow up to this study, as well as trial involving more communities.

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