Counting The Number of Arrangements of Tatami Mats in a Rectangular Room of Vertical Length 2, 3 and 4

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Abstract. Japanese rooms are measured by the number of tatami mats that will fit inside. The size of a tatami mat can vary by region, but is generally around 180 cm by 90 cm, giving it a 2:1 ratio of length to width. In the following, for simplicity, we suppose that each tatami mat is a rectangle with two adjacent sides of lengths 1 and 2. A typical tea ceremony room is square-shaped and its area is the equivalent of 4 and a half tatami mats. Questions regarding to lay tatami mats are not only fun for elementary school students, but also often included in entrance exams. In this paper, we derive recurrence formulae for determining the number of ways to lay tatami mats in a rectangular room whose vertical length is fixed at four or less, by using the concept of *compartments* or indivisible factors. Since the area of each tatami mat is two, if the area of the room is odd, only one half-sized tatami mat is allowed to be used. Therefore, if the vertical length of the room is three, the results will be different depending on whether the horizontal length of the room is even or odd. A generating function is used in this case, since it is difficult to derive the recurrence formula from direct consideration.

1 Introduction

Japanese rooms are measured by the number of tatami mats that will fit inside. The size of a tatami mat can vary by region, but is generally around 180 cm by 90 cm, giving it a 2:1 ratio of length to width. A typical room for tea ceremony is a square-shaped room of size four and a half tatami mats. A popular living room in a house is of size six. For auspicious reason, they are often arranged so that no four mats meet at a point. Students will recognize that a tatami mat is a rectangle and that half a tatami is a square. Half-sized tatami mats are also available only if it is unavoidable to use. For example, Figure 1 shows four types of arranging tatami mats in a square room: an auspicious room (A), a sorrow room in a temple (B), a tea ceremony room (C), and a room for seppuku (D).

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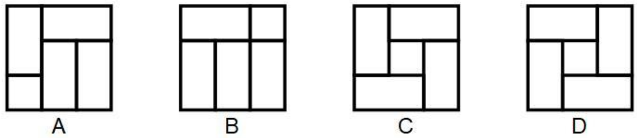

Fig. 1. Types of tatami patterns for a square room of size four and a half.

In the following, for simplicity, we suppose that each tatami mat is a rectangle with two adjacent sides of lengths 1 and 2, and a half-sized tatami mat is a square with a side length of 1. A tatami mat is also called a domino in some literature.

Questions regarding the number of ways to lay tatami mats in a room of given size and shape are not only fun for elementary school students, but often included in entrance exams. For example, the 3rd mathematics problem of early entrance exam for the science students of the University of Tokyo in March 1995 was about the number of ways to cover a room of vertical length 2 and horizontal length n with rectangular mats of size 2 times 1 and square mats of size 2 times 2. Another example is the 2nd part of the 4th question of Mathematics II and Mathematics B of the supplementary examination for the 2021 University Entrance Common Test in Japan was about the ways of covering a rectangular room of vertical length 3 and horizontal length $2n$ with tatami mats.

The mathematical theory of tatami tiling has been studied in detail in recent years, and connections to graph theory, games, and physics are becoming apparent. For more information, consult Read [19], Erickson [4], Alhazov et al [2], Ruskey and Woodcock [20], Kenyon [12] and Erickson et al [5].

In this paper, we derive recurrence formulae for determining the number of ways to lay tatami mats in a rectangular room whose vertical length is fixed at two, three, or four, by using the concept of indivisible factors. We mention that this concept is so natural that it can emerge from discussion in a class of elementary school students of early grades. Since the area of each tatami mat is 2, if the area of the whole room is odd, one must use half-sized tatami mats. In this case, it is interesting to allow to use only one half-sized tatami mat. Therefore, if the room vertical length is fixed to three, the resulting number of tatami arrangements will be somewhat different depending on whether the horizontal length of the room is even or odd. For high school students, it is sometimes convenient and interesting to introduce the concept of generating functions, since it is difficult to derive the recurrence formula from direct geometrical consideration.

This presentation is based on the author's educational practice for 4 years and the goal is to invigorate classrooms and help children better understand the joy of mathematics. Some of the content was presented at annual meetings of the Mathematics Education Society of Japan in the spring of 2022 and 2024 (*Cf.* [24] and [25]).

2 Results and Discussions

In this section, we will discuss three levels of problems regarding counting the number of ways to lay tatami mats in stages. Students are encouraged to use pencil and paper or wooden blocks to simulate the ways they can arrange tatami mats into a room of given size and shape. Throughout this paper, we forget the auspicious condition mentioned in the introduction. Room sizes are given with vertical lengths 2, 3 and 4, and horizontal length n, where n will be an arbitrary integer greater than or equal to zero.

When vertical length is fixed to two and horizontal length n increases as 1, 2, 3 and so on, the number of arrangements varies as 1, 2, 3, 5, 8 and so on to form the Fibonacci sequence f_n (n = 1, 2, 3, …) shown in Figure 2. Students can enjoy constructing the real pattern by wooden blocks, or by drawing pictures on a paper, and then classify them upon discussion.

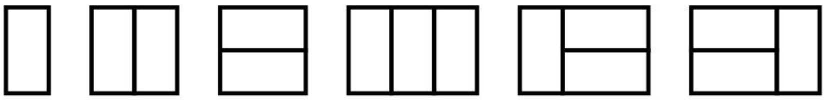

Fig. 2. Some tatami filling patterns for rooms of vertical length two.

After summing up the numbers for several small size rooms, students who can add small whole numbers will recognize the arithmetical law $f_n = f_{n-1}+f_{n-2}$, with $f_1 = 1$ and $f_2 = 2$. Once the increment pattern is obtained, they can calculate as far as they want to get larger digit numbers. This activity, which is intended to the first and second grade elementary school classes, combines the joy of producing new shapes and that of addition to get large numbers.

In my experience, children seem to find intrinsic joy in discovering the Fibonacci numbers. The Fibonacci numbers can also be discovered in other problem settings, such as the number of climbing stairs. Children are influenced by the simplicity and fascination of the discovered patterns, and are motivated to continue adding up considerably far beyond. This activity also meets children's desire to "discover large numbers."

The Fibonacci-type recurrence formula can be derived from the fact that the tatami mats on the left side of the room can be laid out either one vertically or two horizontally, but the younger elementary school students were not observed to discuss this in detail.

Furthermore, if we set $f_0 = 1$, this recurrence formula also holds when $n = 2$. This rule can be interpreted as saying that there is only one way to lay tatami mats in a room of size 2 times 0: to cover it with 0 tatami mats.

By changing the perspective, we can obtain another recurrence formula. Let us classify all the ways in which tatami mats can be laid out in a room of size 2 times n according to the number and position of horizontal tatami mats. If there are no horizontal tatami mats, all the tatami mats are vertical, and there is only one way of laying them out in such a manner. If there are at least one horizontal tatami mats, and we focus on the leftmost one among such tatami mats, then of the n tatami mats, the k-1 tatami mats from the left are vertical, followed by two horizontal tatami mats, and the number of ways of laying them out to the right of those is f_{n-k-1}. This leads to the recurrence formula

$$f_n = 1+ (f_{n-2}+f_{n-3}+\cdots+f_0), \quad (1)$$

where n is greater than or equal to 2.

Next, it is natural to move onto the case where the vertical length is three. When the vertical length is fixed to three, the problem will be more complicated. Indeed, a question to find the recurrence formula in this case was asked in the Common Test for University Admissions in Japan in 2022.

Consider the number b_n of ways to lay tatami mats in a rectangular room of size 3 times n. When n is an odd number, $b_n = 0$. Figure 3 shows that there are just three arrangements for a room of horizontal length 2 (A, B and C in Figure 3), whilst for a room of horizontal length 4, some mat patterns can be divided into right and left parts by a vertical line (D and E), and there is a filling that cannot be divided in such a way (F), which is used in fact in the typical Japanese living room of six tatami mats. For some fixed horizontal length n, students are encouraged to share as many arrangements as possible in a small group and/or the whole class and classify them.

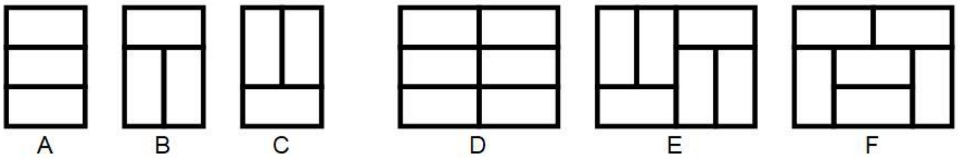

Fig. 3. Some tatami filling for rooms of vertical length three.

An arrangement is called *compartment* if it is impossible to divide by any vertical line. Any arrangement is either a compartment or a combination of more than one compartments. Because there are three compartments of horizontal length two and two compartments of horizontal length four, the number of arrangements of horizontal length four is $3^2+2 = 11$. Here multiplication comes in naturally with an important role. Because there are always 2 compartments of horizontal length n, with n being even and larger than 4, one can calculate that there are $2+2\cdot3\cdot2+2^2 = 41$ arrangements for a room of horizontal length 6, so $b_6 = 41$.

It is possible to continue calculating the number of filling patterns of rooms of vertical length 3 and horizontal length n, with n being even, using the concept of compartments and a recursive formula. First, it is easy to speculate that the number of compartments of any fixed horizontal length greater than 2 is 2. Any layout can be divided into a union of compartments by dividing it with a vertical dividing line. Let us use this fact to find a recurrence formula. Given a layout with horizontal length $n = 2m$, let the horizontal length of the leftmost compartment be $2k$. There are three ways to choose it when $k = 1$, and two ways when k is 2 or more. Thus, by classifying by k being 1, 2, ..., or m, we obtain

$$b_n = 2\,(b_0+b_2+\cdots+b_{n-4}) +3\, b_{n-2}\,, \qquad (2)$$

when n is even. By subtracting this equation from equation

$$b_{n-2} = 2\,(b_0+b_2+\cdots+b_{n-6}) +3\, b_{n-4} \qquad (3)$$

obtained by replacing n with n-2, we obtain the recurrence equation

$$b_n = 4\, b_{n-2} - b_{n-4}\,, \qquad (4)$$

for n 4 or more. Since we only need to consider the case where n is even, this is also a Fibonacci-type three-term recurrence formula, with a general term like Binet's formula. Calculating several values of b_n, we get the following table:

Table 1. Number of arranging tatami mats in a room of vertical length 3.

n	0	2	4	6	8	10
b_n	1	3	11	41	153	571

The author would also like to mention that the fourth question in the supplementary exam of 2021 University Admission Common Test in Japan is a guided-form question that shows another way to derive this recurrence formula.

Similar method is valid for rooms of vertical length four. Let c_n be the number of arranging tatami mats in a rectangular room of size 4 times n, and g_n that of compartments of horizontal length n. Figure 4 shows the complete set of compartments with horizontal length four and five.

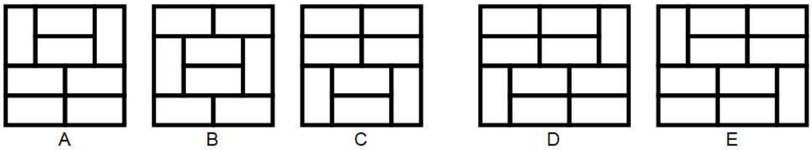

Fig. 4. Typical covering patterns of vertical length four which is vertically undividable.

As shown in Table 2, if the horizontal length n of the room increases as 1, 2, 3, 4, 5, 6 and so on, the number of compartments varies as 1, 4, 2, 3, 2, 3, with 2 and 3 repeated periodically from there on.

Table 2. Number of compartments in a room of vertical length 4.

n	1	2	3	4	5	6	7
g_n	1	4	2	3	2	3	2

This guess is correct, and just like the previous case with vertical length 3, if we focus on the leftmost compartment, we get a set of equations, such as

$$c_4 = c_3 + 4\,c_2 + 2\,c_1 + 3\,c_0, \quad (5)$$
$$c_6 = c_5 + 4\,c_4 + 2\,c_3 + 3\,c_2 + 2\,c_1 + 3\,c_0. \quad (6)$$

Now we get

$$c_n = c_{n-1} + 5\,c_{n-2} + c_{n-3} - c_{n-4} \quad (7)$$

by taking the difference between the equations for c_n and c_{n-2}. After calculating some values, we get Table 3.

Table 3. Number of arranging tatami mats in a room of vertical length 4.

n	0	1	2	3	4	5	6	7
c_n	1	1	5	11	36	95	281	781

The number c_5=95 is the number of arrangements of tatami mats in a room of size 4 times 5. In Japanese, a room of this size is called a 10-tatami room. The question of guessing this number was asked on a mathematics entertainment program "Takeshi's Komadai Mathematics Department" that aired on Fuji TV on Saturday, January 24, 2009.

In general, there is a famous formula for the number of ways to lay tatami mats in a rectangular room of size *m* times *n* that applies the dimer model of statistical mechanics, and it is explained in detail in Kenyon [12].

The method of covering a rectangular room of size 3 times *n*, with *n* odd, using only one half-tatami mat could also be taken up as an interesting theme of mathematical activities at the high school level. We will not discuss this problem here, but details are given in Ueno [26].

3 Conclusions

The general problem for a rectangular room of size *m* times *n* is called the dimer problem and is equivalent to the problem of counting the number of perfect matchings in a grid graph (see for instance Read [1]). Allowing half mats and imposing the auspicious restriction results in visually appealing structure, which is installed in combinatorial games by Alejandro Erickson in 2013 [2].

I would like to thank Hisao Oikawa of Shitennoji University for his heartful discussion. I would also like to thank the primary and secondary school students of the Kunitachi Mathematics Club who actively participated in the discussions during the classes.

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