On Independent [1, 2]-sets in Hypercubes

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Abstract. Given a simple graph G, a subset $S \subseteq V(G)$ is an independent [1, 2]-set if no two vertices in S are adjacent and for every vertex $v \in V(G)\backslash S$, $1 \leq |N(v) \cap S| \leq 2$, that is, every vertex $v \in V(G)\backslash S$ is adjacent to at least one but not more than two vertices in S. This paper investigates the existence of independent [1, 2]-sets of hypercubes. We show that for some positive integer k, if $n = 2^k - 1$, then hypercubes Q_n and Q_{n+1} have an independent [1, 2]-set. Furthermore, for $1 \leq n \leq 4$, we find bounds for the minimum and maximum cardinality of an independent [1, 2]-set of hypercube Q_n, while for $n = 5, 6$, we get the maximum of cardinality of an independent [1, 2]-set of hypercube Q_n.

1 Introduction

Let G be a simple graph, that is, it is an undirected graph, has no loop, and has no multiple edges. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u|uv \in E(G)\}$ of vertices adjacent to v. Each vertex in $u \in N(v)$ is called a *neighbor* of v and the degree of v is $d(v) = |N(v)|$. For a set S and a vertex v, we denote the number of neighbors of v in S as $d_S(v)$, that is, $d\_S\ (v) = |N(v) \cap S\ |$. A set S is *independent* if no two vertices in S are adjacent and *dominating* if every vertex not in S is adjacent to some vertices in S.

Chellali et al., in [1], define a subset $S \subseteq V(G)$ to be a $[j,k]$-set if for every vertex $v \in V(G)\backslash S, j \leq dS\ (v) \leq k$, that is, every vertex in $V(G)\backslash S$ is adjacent to at least j vertices, but not more than k vertices in S. For $j = 1$, a $[1,k]$-set S is a dominating set, since every vertex in $V(G)\backslash S$ has at least one neighbor in S (is dominated by S). The major focus in this study is finding bounds on the minimum cardinality of a [1, 2]-set [1]–[4].

In [5], Chellali et al. continue the study of $[j,k]$-sets and add the requirement that the sets be independent. A dominating set S is an independent $[1,k]$-set of G if S is independent and $1 \leq d_S(v) \leq k$ for all $v \in V(G)\backslash S$. In this paper, we will exclusively focus on independent $[1,2]$-set. Given a graph G, we denote by $i_{[1,2]}(G)$ the minimum cardinality of an independent $[1,2]$-set of G and by $\alpha_{[1,2]}(G)$ the maximum cardinality of an independent $[1,2]$-set of G. Unfortunately, not every graph has an independent $[1,2]$-set. Thus, beside finding the lower and upper bounds cardinality of an independent $[1,2]$-set of a graph, investigating the existence of an independent $[1,2]$-set for some graphs is another focus in this study [6], [7].

In this work, we investigate the existence of independent $[1,2]$-sets of hypercube Q_n. Moreover, we find bounds for the minimum and maximum cardinality of an independent

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$[1,2]$-set of hypercube Q_n, for $n = 1,2,3$, and 4. For $n = 5,6$, we get bounds for the maximum cardinality of an independent $[1,2]$-set of hypercube Q_n.

The n-cube or n-dimensional hypercube Q_n is defined recursively in terms of the Cartesian product of two graphs as follows [8]:

$$Q_1 = K_2 \text{ (a complete graph of order 2)}$$
$$Q_n = K_2 \square Q_{n-1}$$

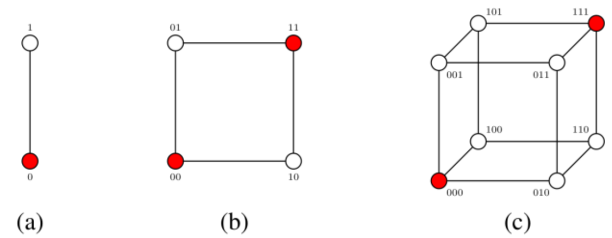

Fig. 1. Hypercubes (a) Q_1, (b) Q_2, and (c) Q_3 with an independent $[1,2]$-set.

The hypercube of dimension n may also be defined as a graph with vertex set $V(Q_n)$ the set of all binary n-tuples of zeros and ones and set $E(Q_n)$ the set of pairs of vertices $u = u_1u_2...u_n$ and $v = v_1v_2...v_n$, where $\sum_{i=1}^{n}|u_i - v_i| = 1$. In this representation, two vertices of Q_n are adjacent if and only if their binary n-tuples differ in exactly one place. **Fig. 1** illustrates hypercubes Q_1, Q_2, and Q_3.

Observation 1 *The sets* $S_1 = \{0\}$ *and* $S_2 = \{00,11\}$ *are independent* $[1,2]$*-sets of* Q_1 *and* Q_2*, respectively. Furthermore,* $i_{[1,2]}(Q_1) = \alpha_{[1,2]}(Q_1) = 1, i_{[1,2]}(Q_2) = \alpha_{[1,2]}(Q_2) = 2$*, and* S_1 *is an efficient dominating set of* Q_1*.*

Observation 2 *The set* $S_3 = \{000,111\}$ *is an independent* $[1,2]$*-set of* Q_3*. No singleton subset of* $V(Q_3)$ *can be a dominating set and any independent set of cardinality three is not a* $[1,2]$*-set; hence* $i_{[1,2]}(Q_3) = \alpha_{[1,2]}(Q_3) = 2$*. Furthermore,* S_3 *is an efficient dominating set of* Q_3*. The independent* $[1,2]$*-set* S_3 *is not unique.*

In Observations 1 and 2 we noted that hypercubes Q_1 and Q_3 have efficient dominating sets. The study of the existence of efficient dominating sets in Q_n has been done in the context of single error-correcting codes [9].

Theorem 1 (Livingston [10]). *An* n*-dimensional hypercube* Q_n *has an efficient dominating set if and only if* $n = 2^k - 1$*, for some positive integer* k*.*

2 Main Result

In the following discussion, we show that hypercubes Q_n, for $n = 4,5$, and 6, have an independent $[1,2]$-set.

Proposition 1. Q_4 *has an independent* $[1,2]$*-set. Furthermore,* $i_{[1,2]}(Q_4) = \alpha_{[1,2]}(Q_4) = 4$*.*

Proof. We prove the proposition by construction. We will construct an independent $[1,2]$-set S_4 using S_3 in Observation 2.

Since by definition, $Q_4 = K_2 \square Q_3$, as illustrated in **Fig. 2**, we may consider Q_4 as formed from two copies of Q_3, say A and B, respectively. We label the vertices of Q_4 using vertex labeling of Q_3 with prefix 0 added in the vertices in A, and 1 for the vertices in B. Thus if $v_1v_2v_3 \in V(Q_3)$, then its corresponding vertices in Q_4 will be $0v_1v_2v_3$ and $1v_1v_2v_3$.

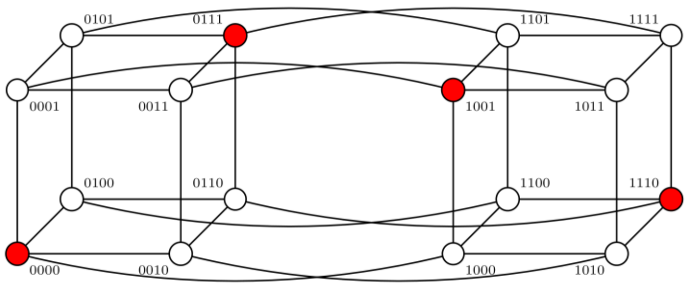

Fig. 2. The hypercubes Q_4 with an independent $[1,2]$-set.

We form S_4 by taking a copy of S_3 in A, and another copy in B. For example, $S_4 = \{0000,0111,1001,1110\}$. It follows that $i_{[1,2]}(Q_4) = 4$ since a vertex subset with at most 3 elements can only dominate 12 vertices at most, while $|V(Q_4)| = 16$. Finally, $\alpha_{[1,2]}(Q_4) = 4$ since an independent $[1,2]$-set with 5 elements will have at least 3 elements in A or B, contradicting $\alpha_{[1,2]}(Q_3) = 2$. □

We note that the independent $[1,2]$-set of Q_4 is not unique. **Fig. 3** shows another independent $[1,2]$-set of Q_4. The advantage of the construction in the proof of Proposition 1 is that we use the independent $[1,2]$-set of Q_3 to construct an independent $[1,2]$-set of Q_4. Using a similar technique, we construct an independent $[1,2]$-set of Q_5 and Q_6. Also, we observe in **Fig. 2**, that S_4 is not an independent $[1,1]$-set. Furthermore, some vertices in Q_4, namely vertices $0001, 0110, 1000, 1111$, are adjacent to two elements of S_4. We need to consider such vertices in the construction of S_5.

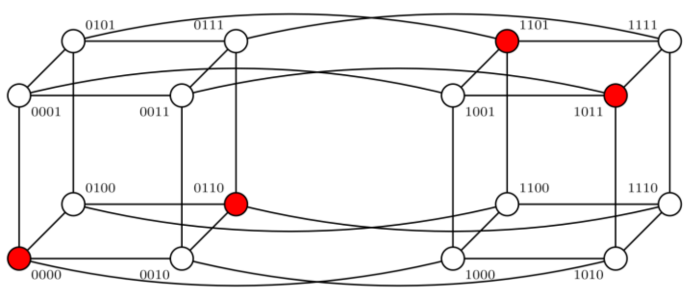

Fig. 3. Another independent $[1,2]$-set for Q_4.

Proposition 2. *Q_5 has an independent $[1,2]$-set and* $\alpha_{[1,2]}(Q_5) = 8$.

Proof. We prove the proposition by construction. We will construct an independent $[1,2]$-set S_5 using S_4 in Proposition 1. We consider Q_5 as formed from two copies of Q_4, say A and B, respectively. We label the vertices of Q_5 using vertex labeling of Q_4 with prefix 0 added in the vertices in A, and 1 for the vertices in B.

As illustrated in **Fig. 4**, we form S_5 by taking a copy of S_4 in Proposition 1 for A. We consider another independent [1,2]-set in B such that the union with S_4 is an independent of Q_5. For example, $S_5 = \{00000,00111,01001,01110,10101,10010,11100,11011\}$. It follows that $\alpha_{[1,2]}(Q_5) = 8$ since an independent $[1,2]$-set with 9 elements will have at least 5 elements in A or B, contradicting $\alpha_{[1,2]}(Q_4) = 4$. □

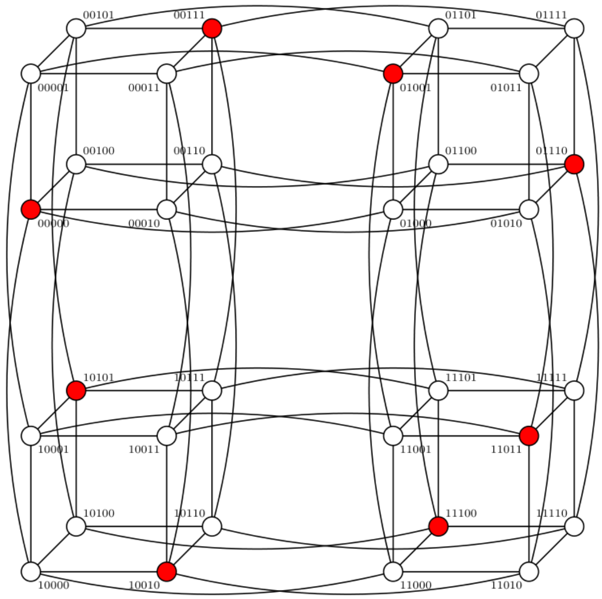

Fig. 4. The hypercube Q_5 with an independent $[1,2]$-set.

Proposition 3. *Q_6 has an independent $[1,2]$-set and $\alpha_{[1,2]}(Q_6) = 16$.*

Proof. Using similar technique as in Propositions 1 and 2, we construct an independent $[1,2]$-set S_6 using S_5. We consider Q_6 as formed from two copies of Q_5, say A and B, respectively. We use S_5 in Proposition 2 as an independent [1,2]-set for A. As illustrated in **Fig. 5**, we construct an independent [1,2]-set for B such that the union with S_5 is an independent [1,2]-set of Q_6.

$$S_6 = \{000000, 000111, 001001, 001110, 010101, 010010, 011100, 011011, 100100, 100011, 101101, 101010, 110001, 110110, 111000, 111111\}$$

is an independent $[1,2]$-set of Q_6. It follows that $\alpha_{[1,2]}(Q_6) = 16$ since an independent $[1,2]$-set with 17 elements will have at least 9 elements in A or B, contradicting $\alpha_{[1,2]}(Q_5) = 8$. □

Theorem 2 If $n = 2^k - 1$, for some positive integer k, then Q_n and Q_{n+1} have an independent $[1,2]$-set.

Proof. By proposition 1, Q_n with $n = 2^k - 1$ has an efficient dominating set, that is, an independent [1, 1]-set, say S_n. Then S_n is also an independent $[1,2]$-set. If $A = \{a_1a_2a_3 \dots a_n a_{n+1} \in V(Q_{n+1}) | a_1 = 0\}$, then $\mathrm{T} = \{0\mathrm{s}_1\mathrm{s}_2\mathrm{s}_3 \dots \mathrm{s}_\mathrm{n} | \mathrm{s}_1\mathrm{s}_2\mathrm{s}_3 \dots \mathrm{s}_\mathrm{n} \in \mathrm{S}_\mathrm{n}\}$ is an independent [1, 1]-set of Q_{n+1}[A]. We observe that

$$W = \{u_1u_2u_3 \dots u_n \in V(Q_n) | u_k = s_k, 1 \leq k \leq n-1, \text{and } |u_n - s_n| = 1, \text{for all } s_1s_2s_3 \dots s_n \in S_n\}$$

is also an independent $[1,1]$-set of Q_n. Let $B = \{b_1b_2b_3 \dots b_n b_{n+1} \in V(Qn+1) \mid b1 = 1\}$. Then a set $U = \{1u_1u_2u_3 \dots u_n | u_1u_2u_3 \dots u_n \in W\}$ is an independent [1, 1]-set of Q_{n+1}[B].

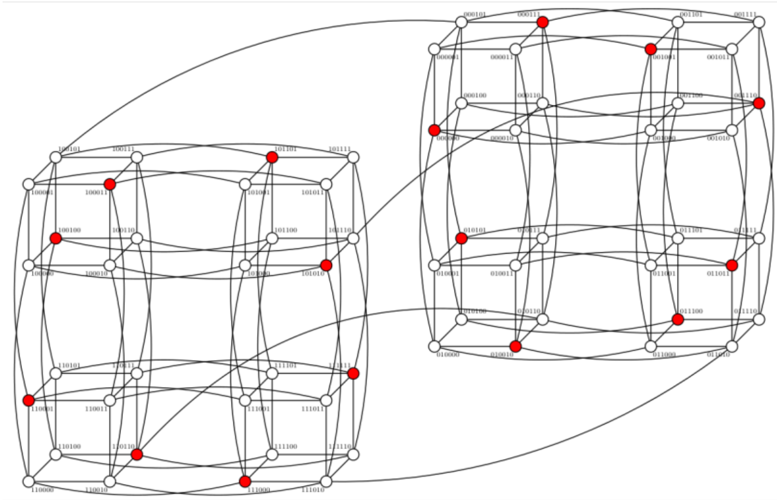

Fig. 5. The hypercube Q_6 with an independent $[1,2]$-set. To avoid a confusion, we only represent some edges from vertices $0v_2v_3v_4v_5v_6$ to $1v_2v_3v_4v_5v_6$.

Now we consider $\mathrm{S_{n+1}} = \mathrm{T} \cup \mathrm{U}$. First we note that S_{n+1} is a dominating set of Q_{n+1} since sets T and U are independent $[1,1]$-sets of $Q_{n+1}[A]$ and $Q_{n+1}[B]$, respectively, where sets A and B are disjoint sets with union $V(Q_{n+1})$. Let $v = v_1v_2v_3\dots v_{n+1}$ and $w = w_1w_2w_3\dots w_{n+1}$ be elements of S_{n+1}. If $v_1 = w_1$ then they are not adjacent since either both of them are elements of T or U.

Suppose $v_1 \neq w_1$. Then one of them is an element of T and the other one is an element of U. We recall the construction of T and U. If $0v_2v_3\dots v_nv_{n+1}$ is an element of T then vertex $1v_2v_3\dots v_nk$ with $k = |v_{n+1} - 1|$ is an element of U. Conversely if $1w_2w_3\dots w_nw_{n+1}$ is an element of U, then vertex $0w_2w_3\dots w_nk$ with $k = |w_{n+1} - 1|$ is an element of T. Thus, if $v_1v_2v_3\dots v_{n+1}$ and $w_1w_2w_3\dots w_{n+1}$ are elements of S_{n+1} with $v_1 \neq w_1$, then $\sum_{i=1}^{n+1}|v_i - w_i| \geq 2$. Hence they are not adjacent.

We have shown that S_{n+1} is an independent dominating set. We need to show that it is a $[1,2]$-set. Let $t \in A$ or $t \in B$. In any case, t is adjacent to at most one element in T and at most one element in U. So, t is adjacent to at most two elements of $S\_\{n+1\}$. By similar argument, if $t\_1 = 1$, then t is adjacent to at most two elements of S_{n+1}, and it follows that S_{n+1} is an independent $[1,2]$-set of Q_{n+1}. □

3 Conclusion and Remarks

This paper has shown the existence of an independent [1, 2]-set of hypercube Q_n for $1 \leq n \leq 6$ and $n = 2k$, for some positive integer k. For further study, one may investigate the existence of an independent $[1,2]$-set of hypercube Q_n for all positive integer n together with the bounds of minimum and maximum cardinality of an independent $[1,2]$-set.

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