Pioneering Numerical Techniques for Solving Differential Equations - A Comprehensive overview

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Abstract. The field of numerical analysis studies the application of mathematics to solve problems of practical importance. When solving differential equations derived from real-world scenarios, numerical techniques play a crucial role, particularly when a closed-form solution is unavailable or obtaining an exact/accurate solution is challenging. This paper's main goal is to look into specific numerical techniques for solving ODEs that have initial conditions. With a primary focus on the Adomian Decomposition, Differential Transform, and Multistep approaches, this study investigates a variety of numerical strategies for solving differential equations. Several mathematicians discovered after a thorough examination of their work that these methods have greatly advanced the analysis of differential equations and are widely used in the fundamental sciences, engineering and economics. The study also emphasizes how essential it is to carry out advanced research in this field so as to create numerical approaches for solving differential equations that are more precise and effective. Research has also carried out on the creation of general-purpose numerical techniques and algorithms for solving the problems, with main focus on stability and convergence in multistep approaches. The two-dimensional nonlinear wave equation is solved using the Adomian Decomposition method, and a unique multistep approach is suggested for handling nonlinear differential equations. The results produced by various techniques are contrasted. **Keywords:** Differential Equations, Numerical Methods, Adomian Decomposition method, Differential Transform Method, Stability, Accuracy. Mathematics Subject Classification (2020): 65L05, 65L06.

1 Introduction

Numerical techniques for explaining differential equations are indispensable tools for solving problems in basic sciences, engineering, and economics. Here's an introduction to some common numerical approaches. Euler's Method is a simple method that estimates the solution by taking small steps along the direction of the slope field. R-K Methods are a family of higher-order methods that use a mixture of function evaluations

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to estimate the solution. In Finite Difference Methods, they separate the differential equation by estimating derivatives with finite differences, then solve the resulting equations. Finite Element Methods divide the problem domain into smaller elements, then use piecewise-defined functions to estimate the solution. Method of Lines will discretize the spatial variables, then solve the subsequent system of ordinary differential equations (ODEs) using numerical techniques. Collocation Methods estimate the solution by finding a function that satisfies the differential equation at a set of collocation points. Each method has its benefits and disadvantages, and the best approach relies on the particular given problem, desired accuracy, and computational efficiency.

Solving ODEs using multistep methods is a general approach in numerical analysis. These methods use the data information from several preceding steps to approximate the result at the succeeding step. Depending on the specific interests and goals, researchers may delve into different aspects of algorithmic development, analysis, and applications of multistep methods in numerical analysis. The DTM is a mathematical technique used for solving differential equations, particularly ODEs and PDEs. It has applications in basic sciences, engineering and finance. It provides an analytical or semi-analytical approach to finding approximate solutions to these equations. DTM was first introduced in 1970s by Ali H. Nayfeh. DTM is predominantly useful when we use nonlinear differential equations or systems of differential equations, where analytical solutions may be tough or dreadful to attain directly. It provides a systematic way to generate approximate solutions that can be used for analysis or engineering applications. However, it's important to note that DTM is an approximation method, and the precision of the results varies with the no. of terms included in series solution. It is best suited for problems where analytical solutions are hard to come by, and numerical methods are not desirable. One effective method for resolving both linear and nonlinear equations is the ADM. The fundamental idea entails breaking down a differential equation's solution into a set of functions.

Multiple numerical techniques were explored by J. C. Butcher [1] who has addressed differential equations using methods such as the implicit Runge-Kutta, Taylor series, hybrid, cyclic composite, and Rosenbrock methods. Butcher discussed the significant progress made in numerical techniques for solving IVPs in ODEs during the 20th century, driven by the advent of electronic computers and the need for efficient numerical algorithms. He explained the evolution of linear multistep methods, particularly the Adams methods, and the R-K methods, which are pivotal in modern computational mathematics. Additionally, J. Peinado et. al [2] evaluated various algorithms for differential Riccati equations, identifying optimal algorithms for both non stiff and stiff cases, emphasizing parameter values, and proposing future work on adaptive algorithms. Samuel O. Adesanya et. Al [3] examined the nonlinear Bratu problem using the Adomian decomposition method, revealing bifurcated and no solutions based on the Frank-Kameneskii parameter. Fayyaz Ahmada [4] proposed a multistep iterative method for nonlinear ODEs and PDEs, showing convergence orders based on the step count.

The ADM addresses the two-dimensional nonlinear wave equation with exceptional accuracy and reliability, providing an analytical solution as a computable series, and identifying self-cancelling 'noise' terms for precision. Comparisons have revealed its effectiveness [5], [6]. The author proposed a technique for solving the Bratu equations using the Laplace ADM and the predictor-corrector technique, demonstrating efficiency and accuracy through comparisons with other methods, particularly at turning and bifurcation points [7]. For Bratu-type equations, the Haar wavelet approach provides great accuracy, simplicity, and low processing costs. It is more accurate and efficient than exact solutions.

For the one-dimensional Bratu equations, Hikmet Caglara et al. [8] introduced a B-spline technique that produced precise and convergent numerical approximations. The accuracy and simplicity of the B-spline method were superior to those of other approaches. For nonlinear singular BVPs, an Integral Decomposition Method was introduced, and it proved accurate and efficient without requiring discretization or linearization [9]. For extremely non-linear initial boundary-value problems, the Adamian Decomposition Method offers exact solutions [10]. The natural decomposition method solves fractional Bratu's IVPs by combining natural transform and ADM for high accuracy [11]. ADM was also developed for linear and nonlinear equations, demonstrating rapid convergence and high accuracy [12].

The Numerov method and implicit 3rd order ND method was compared, with the latter showing better accuracy for solving ODE [13]. A class of multistep methods for 1st order IVPs exhibits superior efficiency and stability [14]. Quintic B-splines and other numerical techniques have been established for fifth-order BVPs [15]. A LMM for 2nd order IVPs has demonstrated good stability [16]. 3rd order IVPs have been solved using numerical differentiation (ND) and differential transform techniques, with the ND method proving to be more successful [17]. It has been demonstrated that a special multistep approach is more accurate than the Galerkin method for solving nonlinear 6th order BVPs [18]. Their study validated the accuracy and stability of a four-step technique for solving 3rd order differential equations.[19].The papers [20] and [21] looked at alternative approaches to differential equation solution, such as the Differential Transform and the Modified Shifted Fractional DTM (MSFDTM) approach (DTM).It was discovered that MSFDTM was a reliable and effective procedure for handling differential equations of fractional order, yielding more accurate solutions than alternative techniques [20]. Linear ordinary differential equations could also be successfully solved by DTM, and the outcomes agreed well with the precise solutions [21].

Comparative research [22] found that the Modified Shifted Fractional Differential Transform Method (MSFDTM) produced more accurate solutions than Numerical Differentiation (ND) for solving 4th order BVPs. They concluded that the differential transformation approach was less precise and less stable than the numerical differentiation methods. Furthermore, the study [23] solved 2-D Bratu equations using the Lie group approach, producing precise solutions that were confirmed by numerical simulations. The study's conclusions broaden our understanding of complex partial differential equations and the fields in which they are used, including combustion theory and biology. Introductory concepts of ADM have been considered from the paper of Man Kwong Mak et al. [24]. Similarly, the approach of DTM is considered from the research paper of C. B. Krishna and S. V. P. Rao [25].

2 Adomian Decomposition Method (ADM)

The ADM is an authoritative technique used to solve nonlinear and linear differential equations. The basic principle involves decomposing the solution of a given problem into several functions of a series. Solutions of nonlinear system of 2nd order differential algebraic equations can be achieved using ADM in a quick and efficient manner.

2.1 Solving 2nd order ODEs via ADM:

Captions should be typed in 9-point Times. They should be centred above the tables and flush left beneath the figures. Consider a 2^{nd} order nonlinear equation as

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + s(x)y + g(x)y^n = k(x) \qquad (1)$$

which are to be addressed along with constraints on y (0) & y'(0), respectively.
Here f, s, g &k are functions of x and n is constant.

$$L^{-1}(*) = \int_0^x e^{-\int f(x)dx} \left(\int e^{\int f(x)dx} \ (*)dx \right)dx \qquad (2)$$

Applying operator L^{-1} , we have

$$L^{-1} \left[\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx}\right] = L^{-1} [k(x) - s(x)y - g(x)y^n$$

$$y(x) = \emptyset(x) + \int_0^x e^{-\int f(x)dx} \left\{\int_0^x e^{\int f(x)dx} \ [k(x) - s(x)y - g(x)y^n]dx\right\} dx$$

Where $\phi(x)$is denoted as

$$\emptyset(x) = y(0) + y'(0) \left[e^{\int f(x)dx}\right] \int_0^x e^{-\int f(x)dx} \ dx \qquad (3)$$

$$\sum_{n=0}^{\infty} y_n(x) = \emptyset(x) + \int_0^x e^{-\int f(x)dx} \ \left[\int_0^x e^{\int f(x)dx} k(x)dx\right]dx \ - \int_0^x e^{-\int f(x)dx} \ \left\{\int_0^x e^{\int f(x)dx} \ [s(x) \sum_{n=0}^{\infty} y_n(x) + g(x) \sum_{n=0}^{\infty} A_n(x)] \ dx \right\} dx$$

Then for the result of the 2^{nd}order equation we have

$$y_0 \ (x) = \emptyset \ (x) + \int_0^x e^{-\int f(x)dx} \ \left[\int_0^x e^{\int f(x)dx}k(x)dx\right] \ dx \qquad (4)$$

$$y_{k+1} \ (x) = -\int_0^x e^{-\int f(x)dx} \ \left[\int_0^x e^{\int f(x)dx} \ [s(x)y_k + g(x)A_k(x)] \ dx \ \right]dx \qquad (5)$$

The result of (1) is $y(x) \ = \sum_{n=0}^{\infty} \ y_n \ (x)$

3 DTM Method

The DTM approach is applied to find the result of ODEs. ODEs were solved using numerical illustrations to demonstrate the strength of the method.
Let y(x) be the function defined in R where x = x_0 denotes a point in R. Then, y(x) is converted as a power series.
For $\frac{d^k}{dx^k}$ y(x), the transform is defined as follows:

$$y(k) \ = \ \frac{1}{k\,!}[\frac{d^k}{dx^k}y(x)]_{x=x_0} \qquad (6)$$

y(k) is transformation of y(x).
Then the solution is $y(x) = \sum_{k=0}^{\infty} x^k Y(k)$
Consider a non-linear 2^{nd} order equation

$$y''(x) = f(x, y(x), y'(x)) \text{ with } y(x_0)= y_0 \text{and } y'(x_0 \ ) = y_1$$

where y′′(x) is the second derivative of y(x) and f is a non-linear function.

Table 1. Transformation of some functions

Function	Transformation
y(x) = f(x)± g(x)	Y(k)= F (k) ±G (k)
y (x)=c f(x)	Y(k)=c F(k)
$y(x) = e^{ax}$	$Y(k) = \frac{a^k}{k!}$
$y(x) = \frac{d^n}{dx^n}\{f(x)\}$	$Y(k) = \frac{(k+n)!}{k!} F(k+n)$
$y(x) = g(x)\ h(x)$	$Y(k) = \sum_{r=0}^{\infty} G(k) F(k-r)$
$y(x) = x^n$	$Y(k) = \delta(k-n)$
$y(x) = sin(ax+b)$	$Y(k) = \frac{a^k}{k!} \sin\left(\frac{k\pi}{2} + b\right)$

The transform of the 2nd derivative y′′(x) is (k+1) (k+2) Y(k+2)
If f (x, y(x), y′(x)) can be articulated in terms of transformed variables, say F(k), then

$$(k+2)(k+1)Y(k+2)=F(k) \tag{7}$$

with y(0)= y_0 and y′(0)= y_1
This gives Y (0) = y_0, Y (1) = y_1
Recurrence Relation:
Substitute k=0,1,2,…..in equation(7) then we obtain Y(k) for different k values.
Then the solution of given equation is
$y(x) = \sum_{k=0}^{\infty} x^k Y(k) = Y(0) + xY(1) + x^2 Y(2) + \ldots\ldots$

4 Multistep Methods:

Multistep methods, such as Adam methods, use multiple previous points to approximate the solution of the equations at the next point.

4.1 General Form:

The general k-step linear approach for addressing y′ = f (x, y) is

$$y_{n+j} = \sum_{j=0}^{k-1} a_j y_{n+j} + h\sum_{j=0}^{k} b_j f_{n+j}$$

Where $a_j and b_j$ are coefficients, h is the step size, and f_{n+j} is the function evaluated at x_{n+j}.
The 2nd order equation

$$y'' = f(x,y),\ y(0) = y_0,\ y'(0) = y_0' \tag{8}$$

commonly occurs in engineering and science.
A k - step linear multistep method for the solution of equation (8) is

$$y_{n+1} = \sum_{j=1}^{k} a_j y_{n+1-j} + h^2 \sum_{j=0}^{k} b_j y''_{n+1-j}$$

Where a’s, b’s are constants & ‘h’ is step size.

4.1.1 Derivation of method:

For the function y(x), p(x) is interpolating polynomial at (k+1) abscissas x_{n+1}, x_n, - - - x_{n-k+1}.

Then $p(x) = \sum_{m=0}^{k}(-1)^m \binom{-s}{m} \nabla^m y_{n+1}$, (9)

Where $s = \frac{(x-x_{n+1})}{h}$

Differentiating (9) twice about x, we have

$$p''(x) = \left(\frac{1}{h^2}\right)\sum_{m=0}^{k} \frac{d^2}{ds^2}\left[(-1^m)\binom{-s}{m}\right]\nabla^m y_{n+1}$$

Substituting $y''(x)$ by $p''(x)$ in equation (9) and keeping x = x_{n+1-r} i.e. taking s = - r, it changes as

$$\sum_{m=0}^{k} \delta_{r,\ m}\nabla^m y_{n+1} = h^2 f_{n+1-r} \quad (10)$$

where $\delta_{r,m} = \frac{d^2}{ds^2}[(-1)^m \binom{-s}{m}]$ (11)

Generating function for $\delta_{r,m}$:

We set to define $D_{r,t} = \sum_{m=0}^{\infty} \delta_{r,m} t^m$ (12)

Substituting $\delta_{r,m}$ from (11) in (12), it modifies as

$$D_{r,t} = \sum_{m=0}^{\infty} \delta_{r,m} t^m = \sum_{m=0}^{\infty}(-t)^m \frac{d^2}{ds^2}\binom{-s}{m}\ at\ s = -r$$

$$\therefore \sum_{m=0}^{\infty} \delta_{r,m} t^m = (1-t)^r[\log(1-t)]^2 ats = -r \quad (13)$$

4.1.2 Explicit methods:

Taking r = 1 in (10), we have a class of methods stated as

$$\sum_{m=0}^{k} \delta_{1,\ m}\nabla^m y_{n+1} = h^2 f_{n+1} \quad (14)$$

From (14), we have $\delta_{1,\ m}$ is coefficient of t^m and $(1-t)^1\ [\log(1-t)]^2$ is written in terms of powers of t.

Table 2 gives the coefficients of $\delta_{1,\ m}$

Deviations in (14) are stated in the form of functional values.

Now (14) will becomes as $\sum_{j=0}^{k} a_j y_{n+1-j} = h^2 f_n$ (15)

a_j's are recorded in table 3.

Truncation error of (15) is provided as $LTE = \delta_{1,\ k+1} h^{k+1} y^{k+1}(\eta)$ (16)

Table 2. Coefficients of $\delta_{1,m}$ $for\ different\ m$

m	0	1	2	3	4	5	6	7
$\delta_{1,\ m}$	0	0	1	0	$\frac{-1}{12}$	$\frac{-1}{12}$	$\frac{-13}{180}$	$\frac{-11}{180}$

Table 3. Coefficients of α_j ; $j = 0\ (1) k,\ k = 2\ (1)\ 7$

K	J							
	0	1	2	3	4	5	6	7
2	1	- 2	1					
3	1	-2	1	0				
4	$\frac{11}{12}$	$\frac{-20}{12}$	$\frac{6}{12}$	$\frac{4}{12}$	$\frac{-1}{12}$			
5	$\frac{10}{12}$	$\frac{-15}{12}$	$\frac{-4}{12}$	$\frac{14}{12}$	$\frac{-6}{12}$	$\frac{1}{12}$		

6	$\frac{137}{180}$	$\frac{-147}{180}$	$\frac{-255}{180}$	$\frac{470}{180}$	$\frac{-285}{180}$	$\frac{93}{180}$	$\frac{-13}{180}$	
7	$\frac{126}{180}$	$\frac{-70}{180}$	$\frac{-486}{180}$	$\frac{855}{180}$	$\frac{-670}{180}$	$\frac{324}{180}$	$\frac{-90}{180}$	$\frac{11}{180}$

The explicit 3rd order ND method
Substitute k=4 in equation (15)

$$\frac{11}{12}y_{n+1} - \frac{20}{12}y_n + \frac{6}{12}y_{n-1} + \frac{4}{12}y_{n-2} - \frac{1}{12}y_{n-3} = h^2 f_n$$

$$11y_{n+1} - 20\,y_n + 6y_{n-1} + 4y_{n-2} - y_{n-3} = 12\,h^2 f_n$$

$$y_{n+1} = \frac{20}{11}y_n - \frac{6}{11}y_{n-1} - \frac{4}{11}y_{n-2} + \frac{1}{11}y_{n-3} + \frac{12}{11}h^2 f_n$$

The 4th order (ND) method k=5 in equation (15)

$$\frac{10}{12}y_{n+1} - \frac{15}{12}y_n - \frac{4}{12}y_{n-1} + \frac{14}{12}y_{n-2} - \frac{6}{12}y_{n-3} + \frac{1}{12}y_{n-4} = h^2 f_n$$

$$10\,y_{n+1} - 15y_n - 4y_{n-1} + 14y_{n-2} - 6y_{n-3} + y_{n-4} = 12h^2 f_n$$

$$y_{n+1} = \frac{15}{10}y_n + \frac{4}{10}y_{n-1} - \frac{14}{10}y_{n-2} + \frac{6}{10}y_{n-3} - \frac{1}{10}y_{n-4} + \frac{12}{10}h^2 f_n$$

The 5th order ND method k=6 in equation (15)

$$\frac{137}{180}y_{n+1} - \frac{147}{180}y_n - \frac{255}{180}y_{n-1} + \frac{470}{180}y_{n-2} - \frac{285}{180}y_{n-3} + \frac{93}{180}y_{n-4} - \frac{11}{180}y_{n-5} = h^2 f_n$$

$$137y_{n+1} - 147\,y_n - 255\,y_{n-1} + 470\,y_{n-2} - 285\,y_{n-3} + 93\,y_{n-4} - 11\,y_{n-5} = 180\,h^2 f_n$$

$$y_{n+1} = \frac{147}{137}y_n + \frac{255}{137}y_{n-1} - \frac{470}{137}y_{n-2} + \frac{285}{137}y_{n-3} - \frac{93}{137}y_{n-4} + \frac{11}{137}y_{n-5} + \frac{180}{137}h^2 f_n$$

In general, multistep approaches solve ODEs with high accuracy requirements more computationally efficiently than single-step methods. They may, however, be more susceptible to beginning circumstances, and the sequence and step size may have an impact on their stability characteristics. To balance accuracy and efficiency, hybrid methods, such as Runge-Kutta methods may combine multistep and single-step techniques for specific situations. The kind of ODE, the required accuracy, and the available computer power all influence which approach is best.

5 Numerical Illustrations:

5.1 Solution by ADM approach:

Let a non-linear 2nd order differential equation be

$$\frac{d^2y}{dx^2} - y = 2e^x, \quad y(0) = 0 \;\&\; y'(0) = -1 \tag{17}$$

Solution of the given equation $\frac{d^2y}{dx^2} - y = 2e^x$ is $y = -e^x + e^{-x} + xe^x$

Equation (17) takes the form of (7) and to be solved with y(0) & y'(0), respectively. Given equation $\frac{d^2y}{dx^2} - y = 2e^x$

Here f(x) = 0, s(x) = -1, g(x) = 0 and k(x)= $2e^x$

$$y_0(x) = \emptyset(x) + \int_0^x e^{-\int f(x)dx}\left[\int_0^x e^{\int f(x)dx} k(x)dx\right]dx$$

From (3) we have $\emptyset(x) = -x$

$$y_0(x) = 2e^x - 3x - 2$$

$$y_{k+1}(x) = \int_0^x\left[\int_0^x y_k(x)dx\right]dx$$

Substitute k=0, we get $y_1(x) = 2e^x - \frac{1}{2}x^3 - x^2 - 2x - 2$

Substitute k=1, then we get

$$y_2(x) = 2e^x - \frac{1}{40}x^5 - \frac{1}{12}x^4 - \frac{1}{3}x^3 - x^2 - 2x - 2$$

Substitute k=2, then we get

$$y_3(x) = 2e^x - \frac{1}{1680}x^7 - \frac{1}{360}x^6 - \frac{1}{60}x^5 - \frac{1}{12}x^4 - \frac{1}{3}x^3 - x^2 - 2x - 2$$

The semi -analytical solution of given equation is

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = 8e^x - \frac{1}{1680}x^7 - \frac{1}{360}x^6 - \frac{1}{24}x^5 - \frac{1}{6}x^4 - \frac{7}{6}x^3 - 3x^2 - 9x - 8 + \ldots$$

Table 4. Solution by ADM for problem1

x	Exact	ADM	Absolute error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	-0.0898164082	-0.0898164082	0.0000000000
0.2	-0.1583914535	-0.1583914534	-0.0000000001
0.3	-0.2040829446	-0.2040829446	0.0000000000
0.4	-0.2247747725	-0.2247747719	-0.0000000006
0.5	-0.2178299756	-0.2178299708	-0.0000000048
0.6	-0.1800358841	-0.1800358597	-0.0000000244
0.7	-0.1075405084	-0.1075404134	-0.0000000950
0.8	0.0042207784	0.0042210864	-0.0000003080
0.9	0.1606093486	0.1606102137	-0.0000008651
1.0	0.3678794412	0.3678816117	-0.0000021705

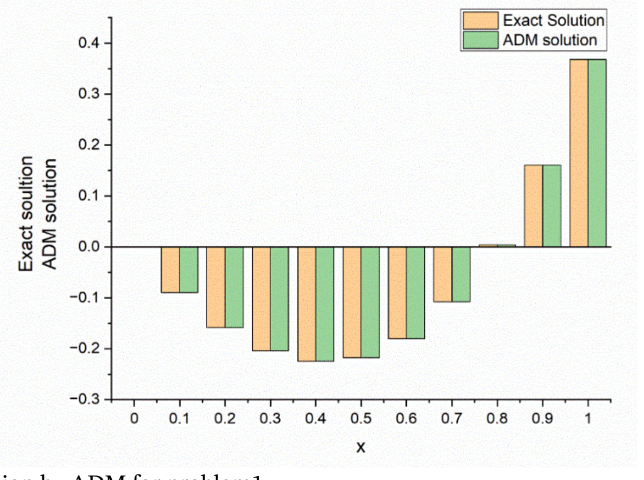

Fig. 1. Solution by ADM for problem1

2.Taking the equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3$, $y(0) = 1$ & $y'(0) = 2$

$y(x) = -e^{-3x} + e^{-x} + 1$ being the solution of the above problem

Here $f(x) = 4$, $s(x) = 3$, $g(x) = 0$ and $k(x) = 3$

$$y_0(x) = \emptyset(x) + \int_0^x e^{-\int f(x)dx} \left[\int_0^x e^{\int f(x)dx} k(x)dx\right] dx$$

$$y_0(x) = \frac{21}{16} + \frac{3}{4}x - \frac{5}{16}e^{-4x}$$

$$y_{k+1}(x) = -\int_0^x e^{-4x}\left[\int_0^x e^{4x} 3\, y_k(x)\, dx\right] dx$$

Substitute k = 0,we get $y_1(x) = -\frac{3}{2}x^2 + x^3 - \frac{3}{8}x^4 - \frac{1}{5}x^5 + \frac{7}{15}x^6$ .....

Substitute K = 1,2,3... $y_{k+1}(x)$ we get

$$y_2(x) = \frac{3x^4}{8} - \frac{9x^5}{20} + \frac{27x^6}{80} - \frac{5x^7}{28} + \frac{9x^8}{140} + \cdots$$

$$y_3(x) = \frac{3x^6}{80} + \frac{3x^7}{56} - \frac{201x^8}{4480} + \frac{23x^9}{840} - \frac{11x^{10}}{840} + \cdots\ldots$$

$$y_4(x) = \frac{9x^8}{4480} - \frac{x^9}{320} + \frac{123x^{10}}{44800} + \cdots$$

The solution of given equation is $y(x) = \sum_{n=0}^{\infty} y_n(x)$

$$y(x) = 1 + 2x - 4x^2 + \frac{13x^3}{3} - \frac{10x^4}{3} + \frac{121x^5}{60} - \frac{91x^6}{90} + \frac{1093x^7}{2520} + \cdots\ldots$$

Table 5. Solution by ADM for problem2

x	Exact	ADM	Absolute error
0.0	1.0000000000	1.0000000000	0
0.1	1.1640191974	1.1640191989	-0.0000000015
0.2	1.2699191170	1.2699195073	-0.0000003903
0.3	1.3342485609	1.3342582568	-0.0000096959
0.4	1.3691258341	1.3692197790	-0.0000939449
0.5	1.3834004996	1.3839440724	-0.0005435728
0.6	1.3835127479	1.3857832686	-0.0022705207
0.7	1.3741288755	1.3817044992	-0.0075756237
0.8	1.3586110108	1.3800577625	-0.0214467517
0.9	1.3393641470	1.3929273904	-0.0535632434
1.0	1.3180923728	1.4392857143	-0.1211933415

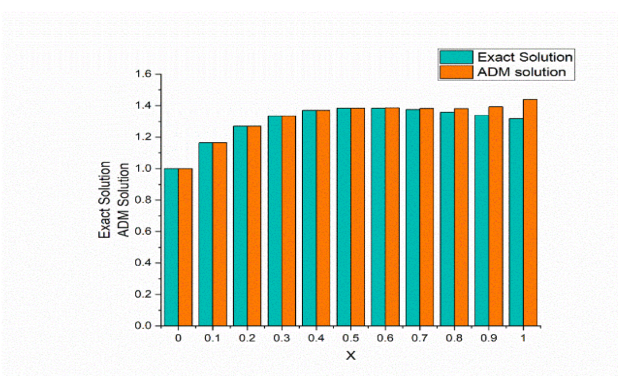

Fig. 2. Solution by ADM for Problem2

5.2 Solution by Differential Transformation Method:

1) $y'' = 2\,e^{x} + y, with \;\; y(0) = 0 \;\; and \;\; y'(0) = -1$

Using the differential transformation, we arrive at

$(k+2)(k+1)\,Y(k+2) = \frac{2}{k!} + Y(k)$ (18)

and $Y(0) = 0,\ Y(1) = -1$ (19)

here transformation of y(x) is Y(k).

Putting (19) in (18) and simplifying, we get

$Y(2) = 1$ $Y(3) = \frac{1}{6}$ $Y(4) = \frac{1}{6}$ $Y(5) = \frac{1}{40}$ $Y(6) = \frac{1}{120}$

$Y(7) = \frac{1}{1007}$ $Y(8) = \frac{1}{5040}$ $Y(9) = \frac{1}{51840}$ $Y(10) = \frac{1}{362880}$ ...

The solution of given equation is

$$y(x) = \sum_{k=0}^{\infty} x^k\, Y(k) = -x + x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{40}x^5 + \frac{1}{120}x^6 + \frac{1}{1007}x^7 + \frac{1}{5040}x^8 + \frac{1}{51840}x^9 + \dots\dots$$

Table 5. Solution by DTM for Problem 1

X	Exact	DTM	Abs. Error
0.0	0.0000000000	0.00000000	0.0000000000
0.1	-0.0898164082	-0.08981640	-0.0000000082
0.2	-0.1583914535	-0.15839145	-0.0000000035
0.3	-0.2040829446	-0.20408294	-0.0000000046
0.4	-0.2247747725	-0.22477477	-0.0000000025
0.5	-0.2178299756	-0.21782997	-0.0000000056
0.6	-0.1800358841	-0.18003588	-0.0000000041
0.7	-0.1075405084	-0.10754051	0.0000000016
0.8	0.0042207784	0.004220757	0.0000000214
0.9	0.1606093486	0.160609270	0.0000000786
1.0	0.3678794412	0.367879188	0.0000002532

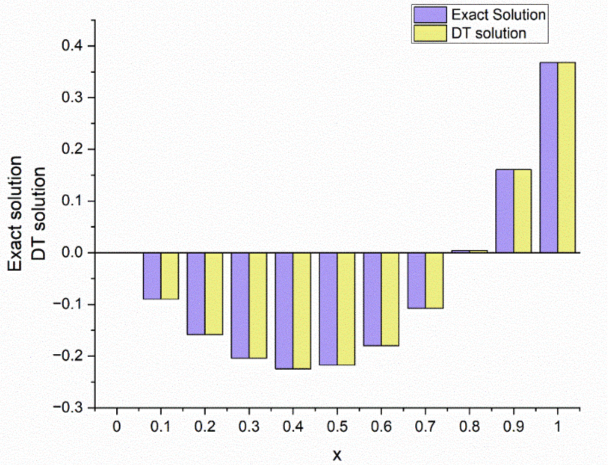

Fig. 3. Solution by DTM for Problem1

2) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3$, y (0) = 1 & y′(0) =2

$y(x) = -e^{-3x} + e^{-x} + 1$ being the solution of the above problem

Taking differential transformation of both sides:

$D\{y''\} + 4D\{y'\} + 3D\{y\} = D\{3\}$

Using the transformation rules

$(k+2)(k+1)Y(k+2)+4(k+1)Y(k+1)+3Y(k)=3\delta(k)$

$\mathrm{y}(0)=1 \Rightarrow \mathrm{Y}(0)=1\ ,y'(0)=2 \Rightarrow Y(1)=2$

For k=0,1,2 …Y(2) $=-4$

$Y(3)=62/6=31/3\ \ \mathrm{Y}(4)=-40/12=-10/3\ \ Y(5)=121/60\ \ Y(6)=-91/90$

The solution of given equation is

$y(x)=\sum_{k=0}^{\infty} x^k\, Y(k)=1+2x-4x^2+\frac{13}{3}x^3-\frac{10}{3}x^4+\frac{121}{60}x^5-\frac{91}{90}x^6$ ------

Table 6. Solution by DTM for Problem 2

x	Exact	DTM	Absolute error
0.0	1.0000000000	1.0000000000	0
0.1	1.1640191974	1.1640191556	-0.0000000015
0.2	1.2699191170	1.2699139556	-0.0000003903
0.3	1.3342485609	1.3341634	-0.0000096959
0.4	1.3691258341	1.3685091556	-0.0000939449
0.5	1.3834004996	1.3805555556	-0.0005435728
0.6	1.3835127479	1.3736416	-0.0022705207
0.7	1.3741288755	1.3459849556	-0.0075756237
0.8	1.3586110108	1.2890979556	-0.0214467517
0.9	1.3393641470	1.1854756	-0.0535632434
1.0	1.3180923728	1.0055555556	-0.1211933415

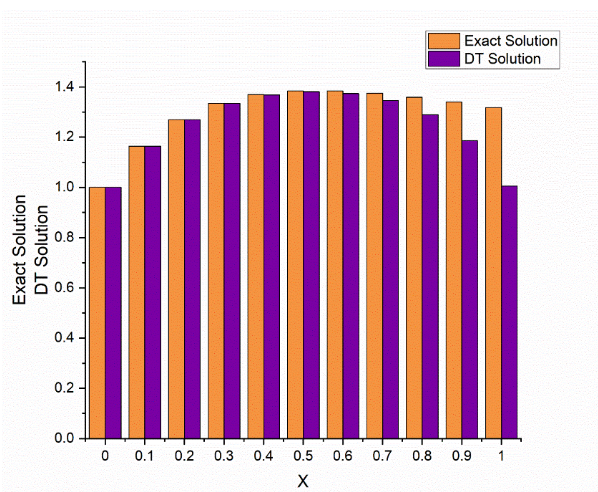

Fig. 4. Solution by DTM for Problem1

5.3 Solution by Multistep Method:

1) $y''=2e^x+y$, with $y(0)=0$ and $y'(0)=-1$

The explicit 4th order numerical differentiation (ND) method formula is

$$y_{n+1} = \frac{15}{10}y_n + \frac{4}{10}y_{n-1} - \frac{14}{10}y_{n-2} + \frac{6}{10}y_{n-3} - \frac{1}{10}y_{n-4} + \frac{12}{10}h^2 f_n$$

Table 7. Solution by Multistep Method for Problem1

X	$Y = -e^x + e^{-x} + xe^x$	Multistep ND Method for k=5	Absolute Error
0.00	0.000000000000	0.109849758000	-0.1098497580
0.1	-0.089816408200	-0.089816408200	0.0000000000
0.2	-0.158391453500	-0.158391453000	-0.0000000005
0.3	-0.204082944600	-0.204082945000	0.0000000004
0.4	-0.224774772500	-0.224774773000	0.0000000005
0.5	-0.217829975600	-0.217829976000	0.0000000004
0.6	-0.180035884100	-0.180035884000	-0.0000000001
0.7	-0.107540508400	-0.107540508000	-0.0000000004
0.8	0.004220778400	0.004220778420	0.0000000000
0.9	0.160609348600	0.160609349000	-0.0000000004
1	0.367879441200	0.367879441000	0.0000000002

Fig. 5. Solution by Multistep Method for Problem1

2) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 3$, with y (0) = 1 and y′(0) =2

The explicit 4th order numerical differentiation (ND) method formula is

$$y_{n+1} = \frac{15}{10}y_n + \frac{4}{10}y_{n-1} - \frac{14}{10}y_{n-2} + \frac{6}{10}y_{n-3} - \frac{1}{10}y_{n-4} + \frac{12}{10}h^2 f_n$$

Table 8. Solution by Multistep Method for Problem2

X	$Y = -e^{-3x} + e^{-x} + 1$	Multistep ND Method for k=5	Absolute Error
0.00	1.0000000000	0.7552344261193820	0.24476557388000
0.1	1.1640191974	1.1640143491017900	0.000004848252454
0.2	1.2699191170	1.2699160622690500	0.000003054714904
0.3	1.3342485609	1.3342467838131200	0.000001777127999
0.4	1.3691258341	1.3691249572193600	0.000000876904077
0.5	1.3834004996	1.3834002477265900	0.000000251837617
0.6	1.3835127479	1.3835129212408300	0.000000173368387
0.7	1.3741288755	1.3741293296548900	0.000000454116467
0.8	1.3586110108	1.3586116419348300	0.000000631107018
0.9	1.3393641470	1.3393648811823000	0.000000734181454
1	1.3180923728	1.3180931579697400	0.000000785166163

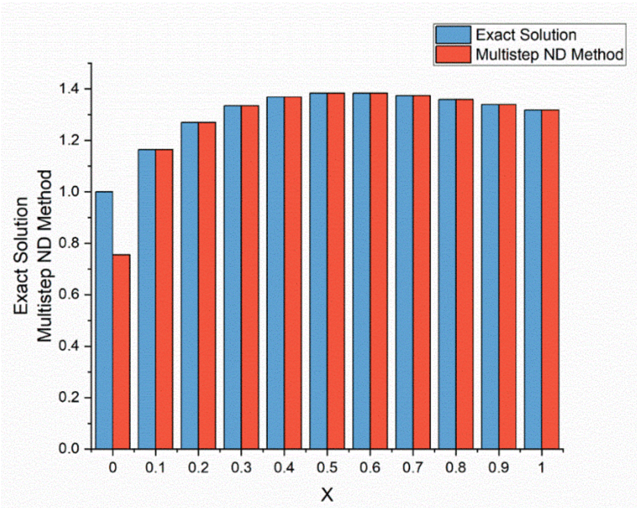

Fig. 6. Solution by Multistep Method for Problem2

6 Comparison of Methods:

We have demonstrated in this work that Adomian Decomposition and Differential Transform methods may be effectively used and a comparison of the obtained findings with the exact solutions is made. It is observed that the ADM and DTM's absolute error is very little, as the visual comparisons make abundantly plain. Non-linear differential equations can now be effectively used to solve practical applications.

Table 9. Comparison of Methods for Problem 1

X	Exact	ADM	DTM	Multistep ND Method
0	0.000000000000	0.0000000000	0.0000000000	0.109849758000
0.1	-0.089816408200	-0.0898164082	-0.0898164	-0.089816408200
0.2	-0.158391453500	-0.1583914534	-0.15839145	-0.158391453000
0.3	-0.204082944600	-0.2040829446	-0.20408294	-0.204082945000
0.4	-0.224774772500	-0.2247747719	-0.22477477	-0.224774773000
0.5	-0.217829975600	-0.2178299708	-0.21782997	-0.217829976000
0.6	-0.180035884100	-0.1800358597	-0.18003588	-0.180035884000
0.7	-0.107540508400	-0.1075404134	-0.10754051	-0.107540508000
0.8	0.004220778400	0.0042210864	0.004220757	0.004220778420
0.9	0.160609348600	0.1606102137	0.16060927	0.160609349000
1	0.367879441200	0.3678816117	0.367879188	0.367879441000

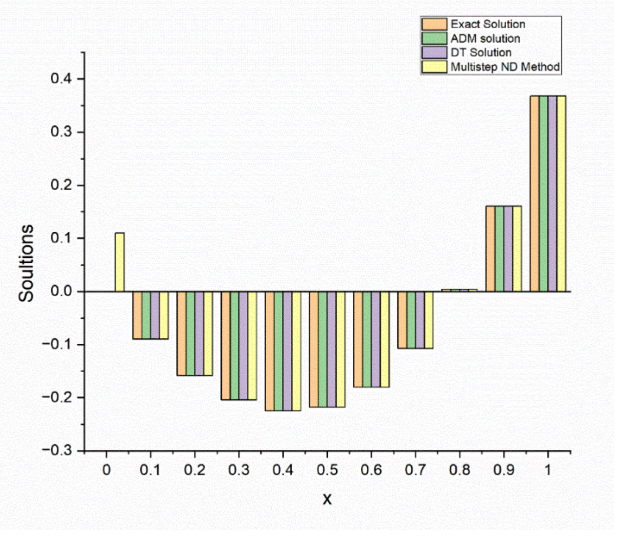

Fig. 7. Comparison of methods for problem 1

The graph displays the comparison between the Exact Solution, ADM Solution, DT Solution, and Multistep ND Method. Exact Solution serves as the reference for accuracy .ADM Solution follows the Exact Solution relatively closely. Deviations from the Exact Solution are minor, indicating a good approximation. Similar to the ADM Solution, the DT Solution closely follows the Exact Solution. There are slight deviations, but they remain small, indicating another good approximation. Multistep method shows significant deviation at the initial point but then aligns well with the exact solution after the initial dip.

Table 10. Comparison of Methods for Problem 2

X	Exact	ADM	DTM	Multistep ND Method
0	1.0000000000	1.0000000000	1.0	0.7552344261193820
0.1	1.1640191974	1.1640191989	1.1640191556	1.1640143491017900
0.2	1.2699191170	1.2699195073	1.2699139556	1.2699160622690500
0.3	1.3342485609	1.3342582568	1.3341634	1.3342467838131200
0.4	1.3691258341	1.3692197790	1.3685091556	1.3691249572193600
0.5	1.3834004996	1.3839440724	1.3805555556	1.3834002477265900
0.6	1.3835127479	1.3857832686	1.3736416	1.3835129212408300
0.7	1.3741288755	1.3817044992	1.3459849556	1.3741293296548900
0.8	1.3586110108	1.3800577625	1.2890979556	1.3586116419348300
0.9	1.3393641470	1.3929273904	1.1854756	1.3393648811823000
1	1.3180923728	1.4392857143	1.0055555556	1.3180931579697400

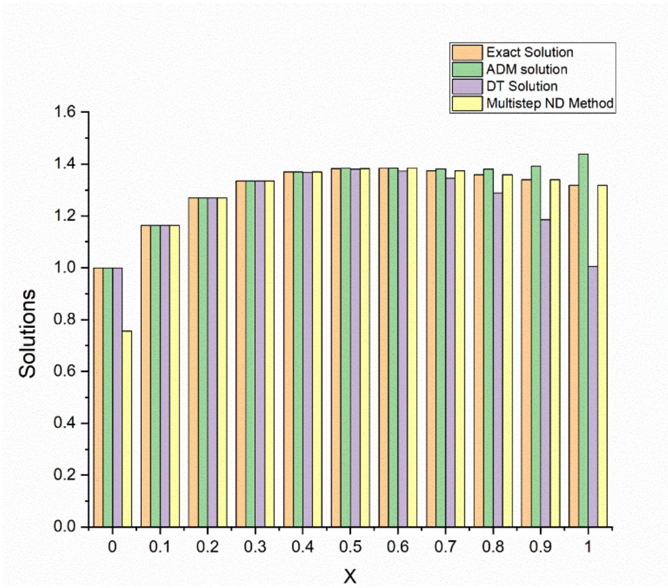

Fig. 8. Comparison of methods for problem 2

The graph compares the accuracy of different numerical methods (ADM, DT, and Multistep ND) against the exact solution. Based on the graph, the Multistep ND method's solution appears to be the most accurate, closely following the exact solution. Except at the initial point, solution by Multistep ND method aligns with the exact solution at all the other points. The ADM solution is also relatively accurate, but there is a slight deviation from the exact solution at the ending points of the interval. Solution by DT method is coinciding in the first half of the interval points, but then deviating in the second half with a noticeable difference from the exact solution

7 Conclusion:

Mathematicians, including Butcher, who contributed to the advancement of numerical approaches for ODEs, have played a vital role in advancing this field. The fundamental techniques discussed in this study serve as a foundation for future innovation, and are expected to result in more efficient and accurate solutions for intricate differential equations. In turn, this will enhance the practical application of these equations across a wide range of scientific and engineering disciplines.

The ADM Solution and DT Solution are both very close to the exact solution, with ADM Solution being slightly more accurate overall. The Multistep ND Method has an initial discrepancy but converges to follow the exact solution closely as x increases. Thus, the ADM Solution is the nearest to the exact solution, followed closely by the DT Solution. The graph clearly indicates that both ADM and DT Solutions provide better approximations than the Multistep ND Method. Multistep method is found close to exact solution for the second problem, whereas ADM and DT methods have fumbled to come near to exact at the later half of interval points. Thus Multistep, Adomian Decomposition and Differential Transform methods are almost running together with respect to different problems.

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