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Dynamics of the Double-Well Duffing System

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Abstract. We consider a Duffing system with double-well potential beam system by F. C. Moon and P. J. Holmes (1979). When there is no forcing, we study the stability of the system with and without small damping. Furthermore, the solution of the unforced system will always be bounded. When the system is perturbed with a small periodic forcing, superharmonic and subharmonic cases will appear. We obtain approximations for the solution in the nonharmonic, superharmonic, and subharmonic cases, which can provide the dynamics of the solution.

1 Introduction

In 1979, Moon and Holmes [1] studied the so-called Duffing equation [2] as a model for vibrations of a beam under the influence of a magnetic field created by two magnets. See Figure 1. The model they considered is given by

$$\ddot{x} + \delta \dot{x} - x + x^3 = \gamma \cos \omega t, \text{ with } \delta, \gamma \geq 0, \omega > 0.$$

Here, δ is the damping parameter. The model also includes external periodic forcing with frequency ω and nonlinear restoring forces.

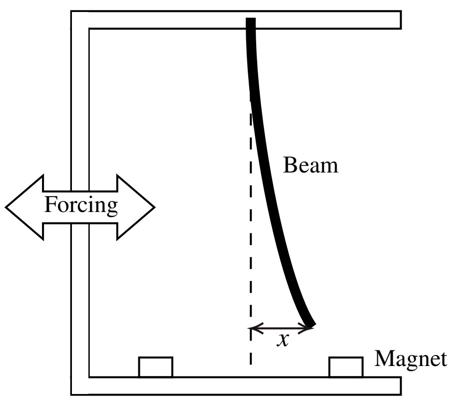

Figure 1. The beam model

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The Duffing equation has emerged as a fundamental tool in the modeling of nonlinear systems. Salas *et al.* (2021) [3] emphasized its significance in capturing nonlinear wave phenomena. J. Sunday (2017) [4] created computational techniques for simulating the Duffing oscillator, showcasing its wide-ranging applications in both science and engineering.
This study explores the dynamics of the double-well Duffing system. By examining the stability of the equilibrium, the existence of bounded solutions, and the resonance behaviors in the presence of damping and external forces, we aim to enhance the understanding of its nonlinear characteristics and expand its potential applications.

2 The system without forcing term

Without damping and forcing, the model can be described by the double-well potential Hamiltonian system which is derived from Newton's law. The minima of the potential correspond to the stable equilibrium of the system; there are two such points. The local maxima at the origin correspond to the saddle-type equilibrium where the beam is in the vertical position.

2.1 The system without damping

Without damping and forcing, the equation of motion reads:

$$\ddot{x} - x + x^3 = 0. \tag{1}$$

Here the time variable is τ and the dot represents the derivative with respect to τ. This equation can be written as a system of first order differential equations:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_1 - x_1{}^3, \end{cases}$$

by setting $x_1 = x$ and $x_2 = \dot{x}$. There are three equilibria in the system, i.e. $(0, 0)$, $(-1, 0)$ and $(1, 0)$. From linear stability analysis, we derive that the first is of saddle type, while the latter are of center type.
To conclude global stability, we look at the first integral (or in our case Hamiltonian function) for equation (1), i.e.

$$\frac{1}{2}x_2{}^2 - \frac{1}{2}x_1{}^2 + \frac{1}{4}x_1{}^4 = E.$$

The Taylor series for E, around $(0, 0)$, $(-1, 0)$ and $(1, 0)$ are:

$$E = \frac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \dots,$$
$$E = \frac{1}{4} + \frac{1}{2}\begin{pmatrix} x_1 + 1 & x_2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 + 1 \\ x_2 \end{pmatrix} + \dots, \text{ and}$$
$$E = \frac{1}{4} + \frac{1}{2}\begin{pmatrix} x_1 - 1 & x_2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} + \dots,$$

respectively. In each case, the quadratic form is represented by a non-degenerate matrix. By Morse Lemma [5] we conclude that the local (linear) stability extends to the global stability.

The first integral is symmetric with respect to the x-axis and to the y-axis. This implies that the level sets of E can be constructed from the information from the first quadrant of the coordinate plane, and mirror-image it to the other quadrants using the axis.
Taking intersection between the level set $E = C$ with $x_2 = 0$ we have:

$$C = -\frac{1}{2}{x_2}^2 + \frac{1}{4}{x_1}^4.$$

This is a quadratic equation in ${x_1}^2$ with discriminant: $C + \frac{1}{4}$. Then, for $C > -\frac{1}{4}$ we always have two solutions for ${x_1}^2$, i.e.:

$${x_1}^2 = 2 \pm 2\sqrt{1 + 4C}.$$

For $C > 0$, we have only one positive root for x_1, i.e.:

$$x_1 = \sqrt{2 + 2\sqrt{1 + 4C}},$$

while for $-\frac{1}{4} < C \leq 0$ we have two solutions:

$$x_1 = \sqrt{2 \pm 2\sqrt{1 + 4C}}$$

Doing similar analysis for the intersection with x_2-axis, we derive that we have a unique intersection at $(0, \sqrt{2C})$, for $C \geq 0$.
Let us now look at the derivatives:

$$\partial_{x_1} E = -x_1\left(1 - {x_1}^2\right) \text{ and } \partial_{x_2} E = x_2.$$

We conclude that $\partial_{x_1} E = 0$ at $x_1 = 1$ for all x_2 while $\partial_{x_2} E = 0$ at $x_2 = 0$ for all x_1. Then both $\frac{dx_2}{dx_1}$ and $\frac{dx_1}{dx_2}$ exist if $x_1 \neq 1$ or $x_2 = 0$. At $x_1 = 1$ we have $\frac{dx_2}{dx_1} = 0$ while at $x_2 = 0$ we have $\frac{dx_1}{dx_2} = 0$. Note that $\partial_{x_1} E = 0$ at $x_1 = 0$ for all $x_2 \neq 0$. We conclude the following.

Theorem 1. *For $C > 0$, the curve*

$$\frac{1}{2}{x_2}^2 - \frac{1}{2}{x_1}^2 + \frac{1}{4}{x_1}^4 = C$$

intersects the x_2-axis twice and the x_1-axis twice. The curve also has four points where $x_1 = \pm 1$ and two points where $x_1 = 0$, at which the tangents of the curve are horizontal. Additionally, there are two points where $x_2 = 0$, at which the tangents of the curve are vertical. Therefore, the curve for $C > 0$ is a closed curve.
For $C = 0$, the curve intersects the x_1-axis at three points, one of which is the origin. There are four points where the tangents are horizontal, specifically at $x_1 = \pm 1$, and two points where the tangents are vertical at $x_2 = 0$. Since the curve contains the origin, which is a saddle point, it consists of two homoclinic loops connecting the saddle point to itself.
By a similar analysis, for $-\frac{1}{4} < C < 0$, the curve consists of two leaves, each forming a closed loop around the center point.

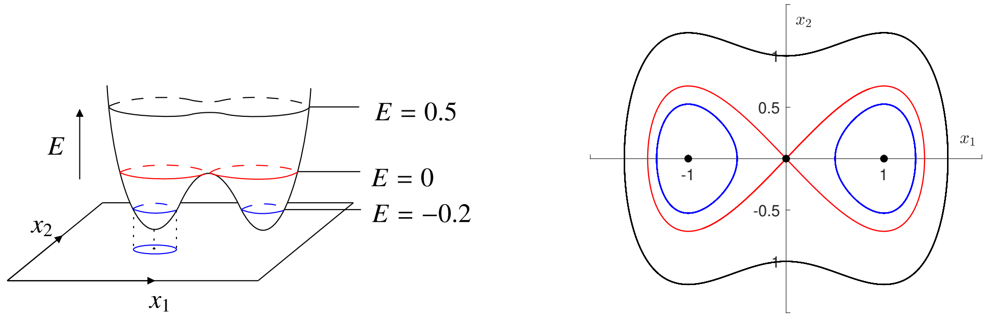


Figure 2. Level set of first integral (left) and the corresponding solution in the phase plane (right)

2.2 The system with damping included

If we include damping, the system becomes

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = x_1 - {x_1}^3 - \delta x_2.$$

The equilibria persist if we include damping. However, the stability of $(\pm 1, 0)$ is now asymptotically stable (focus). Now we want to see how the direction of the solution changes along the level set of E by taking the orbital derivative of E from the previous system. The orbital derivative is

$$\begin{aligned} L_\tau E &= \partial_{x_1} E \dot{x}_1 + \partial_{x_2} E \dot{x}_2 \\ &= (-x_1 + {x_1}^3) x_2 + x_2 (x_1 - {x_1}^3 - \delta x_2) \\ &= -\delta {x_2}^2 \leq 0. \end{aligned}$$

Having the orbital derivative less than or equal to zero means that the solution of the system will move towards the interior of the level curve i.e. one of the equilibria. Therefore, the solution of this system is bounded.

3 System with Small Damping and Forcing

Now we consider a small damping and forcing on the system, $\bar{\delta}, \bar{\gamma} \sim O(1)$,

$$\ddot{x} + \varepsilon \bar{\delta} \dot{x} - x + x^3 = \varepsilon \bar{\gamma} \cos \bar{\omega} \tau.$$

We will specifically look at the neighborhood of the equilibrium $(1, 0)$. Translating the equilibrium to the origin and rescaling x, i.e. t$x = 1 + \varepsilon y$, we derive

$$\ddot{y} + 2y + \varepsilon(\bar{\delta}\dot{y} + 3y^2) + \varepsilon^2 y^3 = \bar{\gamma} \cos \bar{\omega} t,$$

with $y(0) = 0$ and $y'(0) = 0$. We also rescale time by $\tau = \bar{\omega} t$ to fix the period of the forcing term to 2π. As a consequence, we will have an extra parameter ω in the system, i.e.

$$y'' + 2\omega^2 y + \varepsilon(\delta y' + 3\omega^2 y^2) + \varepsilon^2 \omega^2 y^3 = \gamma \cos t, \quad y(0) = 0 \text{ and } y'(0) = 0 \tag{2}$$

with $\omega = \bar{\omega}^{-1}, \delta = \bar{\delta}\omega$, and $\gamma = \bar{\gamma}\omega^2$ are all $O(1)$.

3.1 Two Time Scale Method

We will be using two time scale method [6] to approximate the solution. Let $t_1 = t$ and $t_2 = \varepsilon t$, such that,

$$\frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2},$$

assuming that we can expand the approximate solution y as,

$$y(t_1, t_2) = y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + O(\varepsilon^2).$$

Substituting them to (2), we have

$$\begin{aligned}(\partial_{t_1}^2 + 2\varepsilon\partial_{t_1}\partial_{t_2} + \varepsilon^2\partial_{t_2}^2)(y_0 + \varepsilon y_1) + 2\omega^2(y_0 + \varepsilon y_1)\\ +\varepsilon\left[\delta(\partial_{t_1} + \varepsilon\partial_{t_2})(y_0 + \varepsilon y_1) + 3\omega^2(y_0 + \varepsilon y_1)^2\right]\\ +\varepsilon^2\omega^2(y_0 + \varepsilon y_1)^3 + O(\varepsilon^2) = \gamma\cos t,\end{aligned}$$

with initial value $y = 0$, $(\partial_{t_1} + \varepsilon\partial_{t_2})y = 0$ at $t_1 = t_2 = 0$. We get the equations for $O(1)$ and $O(\varepsilon)$,

$$\begin{aligned}O(1) &: \partial_{t_1}^2 y_0 + 2\omega^2 y_0 = \gamma\cos t_1,\ y_0(0,0) = 0,\ \partial_{t_1} y_0|_{t_1=t_2=0} = 0.\\ O(\varepsilon) &: \partial_{t_1}^2 y_1 + 2\omega^2 y_1 = -2\partial_{t_1}\partial_{t_2} y_0 - \delta\partial_{t_1} y_0 - 3\omega^2 {y_0}^2\\ &\quad \text{with } y_1 = 0,\ \partial_{t_1} y_1 + \partial_{t_2} y_0 = 0,\ \text{at } t_1 = t_2 = 0.\end{aligned}$$

Remark. We will assume that $\omega \neq 1/\sqrt{2}$ because otherwise we will be dealing with secular terms in $O(1)$.
The general solution of y_0 is given by,

$$y_0(t_1, t_2) = a(t_2)\cos(\sqrt{2}\omega t_1) + b(t_2)\sin(\sqrt{2}\omega t_1) + \frac{\gamma}{2\omega^2 - 1}\cos t_1.$$

Substituting the initial value, we get the initial value for a and b,

$$a(0) = -\frac{\gamma}{2\omega^2 - 1}, \quad \text{and} \quad b(0) = 0. \tag{3}$$

If we substitute y_0 to $O(\varepsilon)$ equation, we get

$$\begin{aligned}\partial_{t_1}^2 y_1 + 2\omega^2 y_1 = \left(-2\sqrt{2}\omega b' - \delta\sqrt{2}\omega b\right)\cos(\sqrt{2}\omega t_1) + \left(2\sqrt{2}\omega a' + \delta\sqrt{2}\omega a\right)\sin(\sqrt{2}\omega t_1)\\ + \frac{\delta\gamma}{2\omega^2 - 1}\sin t_1 - 3\omega^2 {y_0}^2,\end{aligned}$$

where,

$$\begin{aligned}{y_0}^2 = &\left(\frac{a^2}{2} + \frac{b^2}{2} + \frac{1}{2}\left(\frac{\gamma}{2\omega^2 - 1}\right)^2\right)\\ &+ \frac{1}{2}\left(\frac{\gamma}{2\omega^2 - 1}\right)^2\cos(2t_1) + \left(\frac{a^2}{2} - \frac{b^2}{2}\right)\cos(2\sqrt{2}\omega t_1) + ab\sin(2\sqrt{2}\omega t_1)\\ &+ \frac{a\gamma}{2\omega^2 - 1}\left(\cos((1 + \sqrt{2}\omega)t_1) + \cos((1 - \sqrt{2}\omega)t_1)\right)\\ &+ \frac{b\gamma}{2\omega^2 - 1}\left(\sin((1 + \sqrt{2}\omega)t_1) - \sin((1 - \sqrt{2}\omega)t_1)\right).\end{aligned}$$

We can see that there are cases where ${y_0}^2$ affects the secular term, where

$$\sqrt{2}\omega = 2 \quad \Leftrightarrow \omega = \sqrt{2} \quad \text{(Superharmonic)},$$
$$\sqrt{2}\omega = 1 - \sqrt{2}\omega \quad \Leftrightarrow \omega = \frac{1}{2\sqrt{2}} \quad \text{(Subharmonic)}.$$

In this paper, we will consider the nonresonance case and the subharmonic case.

3.2 Nonresonance ($\omega \neq \sqrt{2}$ and $\omega \neq \frac{1}{2\sqrt{2}}$)

In this case, the expression ${y_0}^2$ does not contain the terms $\sin(\sqrt{2}\omega t_1)$ and $\cos(\sqrt{2}\omega t_1)$. Since ${y_0}^2$ has no effect on the secular terms in the $O(\varepsilon)$ equation, it can be ignored. From this equation,

$$\partial_{t_1}^2 y_1 + 2\omega^2 y_1 = \left(-2\sqrt{2}\omega b' - \delta\sqrt{2}\omega b\right)\cos(\sqrt{2}\omega t_1) + \left(2\sqrt{2}\omega a' + \delta\sqrt{2}\omega a\right)\sin(\sqrt{2}\omega t_1) + \frac{\delta\gamma}{2\omega^2 - 1}\sin t_1 - 3\omega^2 {y_0}^2.$$

we get $a' = -\frac{\delta}{2}a$ and $b' = -\frac{\delta}{2}b$ with initial condition (3). Solving these, we have

$$a(t_2) = -\frac{\gamma}{2\omega^2 - 1}e^{-\frac{\delta}{2}t_2} \quad \text{and} \quad b(t_2) = 0.$$

Substituting a and b to y_0, the approximate solution is

$$y_0(t_1, t_2) = -\frac{\gamma}{2\omega^2 - 1}e^{-\frac{\delta}{2}t_2}\cos(\sqrt{2}\omega t_1) + \frac{\gamma}{2\omega^2 - 1}\cos t_1,$$

or

$$y(t) \sim -\frac{\gamma}{2\omega^2 - 1}e^{-\frac{\delta}{2}\varepsilon t}\cos(\sqrt{2}\omega t) + \frac{\gamma}{2\omega^2 - 1}\cos t. \tag{4}$$

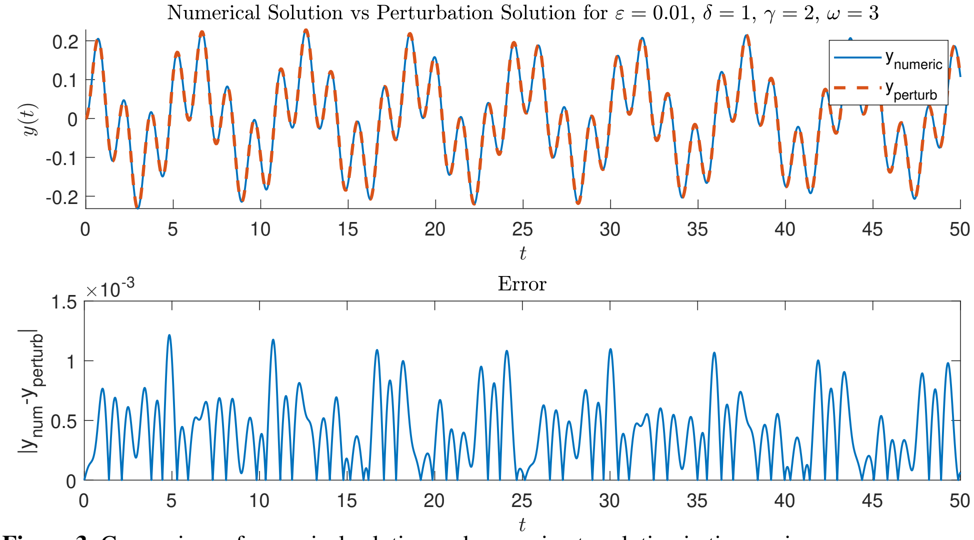


Figure 3. Comparison of numerical solution and approximate solution in time series

The error between the numerical and approximate solution is relatively small compared to ε chosen. Moreover, the solution (4) will leave $\cos(t)$ term for a large t. Hence, the solution (4) is bounded.

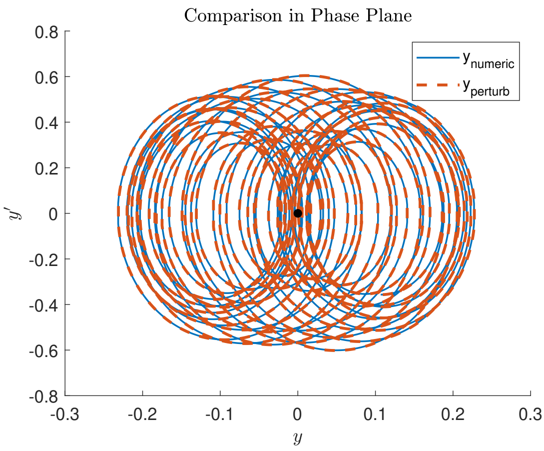


Figure 4. Comparison of numerical solution and approximate solution in phase plane

3.3 Subharmonic Case ($\omega = \frac{1}{2\sqrt{2}}$)

In the subharmonic case, the terms of ${y_0}^2$ that affect the secular term in $O(\varepsilon)$ equation are

$$ {y_0}^2 = -\frac{4\gamma}{3}a\cos\left(\frac{t_1}{2}\right) + \frac{4\gamma}{3}b\sin\left(\frac{t_1}{2}\right) + \text{n.r.} $$

where n.r. are the nonresonant terms. Thus, the $O(\varepsilon)$ becomes

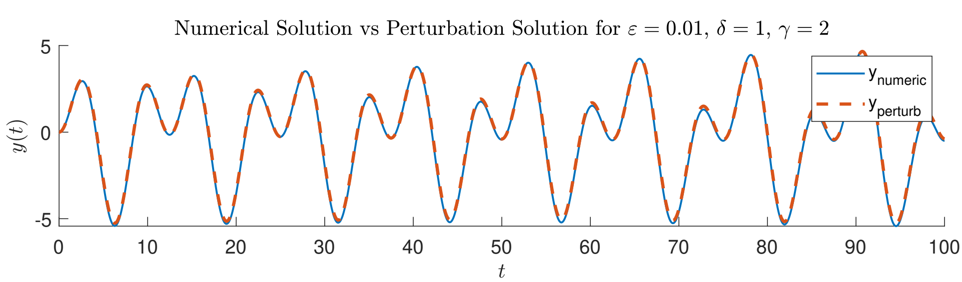
$$ \partial_{t_1}^2 y_1 + \frac{1}{4}y_1 = \left(-b' - \frac{\delta}{2}b + \frac{\gamma}{2}a\right)\cos\left(\frac{t_1}{2}\right) + \left(a' + \frac{\delta}{2}a - \frac{\gamma}{2}b\right)\sin\left(\frac{t_1}{2}\right) + \text{n.r.} $$

From this equation, we have a coupled linear system of a and b. Solving the system with the initial condition (3) substituted $\omega = \frac{1}{2\sqrt{2}}$ gives the approximate solution,

$$ \begin{aligned} y(t) \sim & \left(\frac{2\gamma}{3}e^{\left(\frac{\gamma-\delta}{2}\right)\varepsilon t} + \frac{2\gamma}{3}e^{\left(\frac{-\gamma-\delta}{2}\right)\varepsilon t}\right)\cos\left(\frac{t}{2}\right) \\ & + \left(\frac{2\gamma}{3}e^{\left(\frac{\gamma-\delta}{2}\right)\varepsilon t} - \frac{2\gamma}{3}e^{\left(\frac{-\gamma-\delta}{2}\right)\varepsilon t}\right)\sin\left(\frac{t}{2}\right) - \frac{4\gamma}{3}\cos t. \end{aligned} $$

In this solution, we have the $\exp\left\{\left(\frac{\gamma-\delta}{2}\right)\varepsilon t\right\}$. This term grows unbounded when $\delta < \gamma$. Otherwise, it will be 1 if $\delta = \gamma$ and approaching 0 if $\delta > \gamma$. Thus, this solution is bounded when $\delta \geq \gamma$.
Now, we will look at the simulation for each subcase.

Case 1: $\delta < \gamma$



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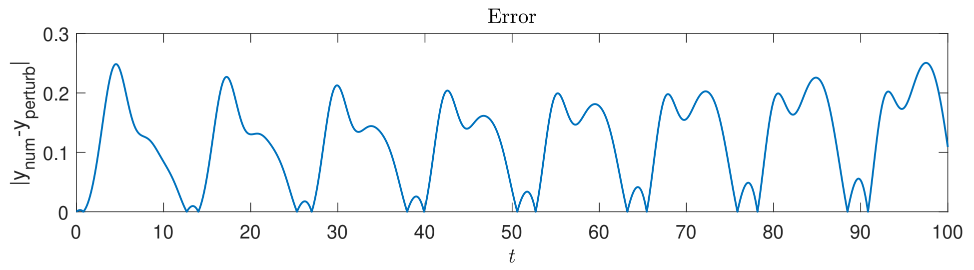

Figure 5. Comparison of numerical solution and approximate solution ($\delta < \gamma$)

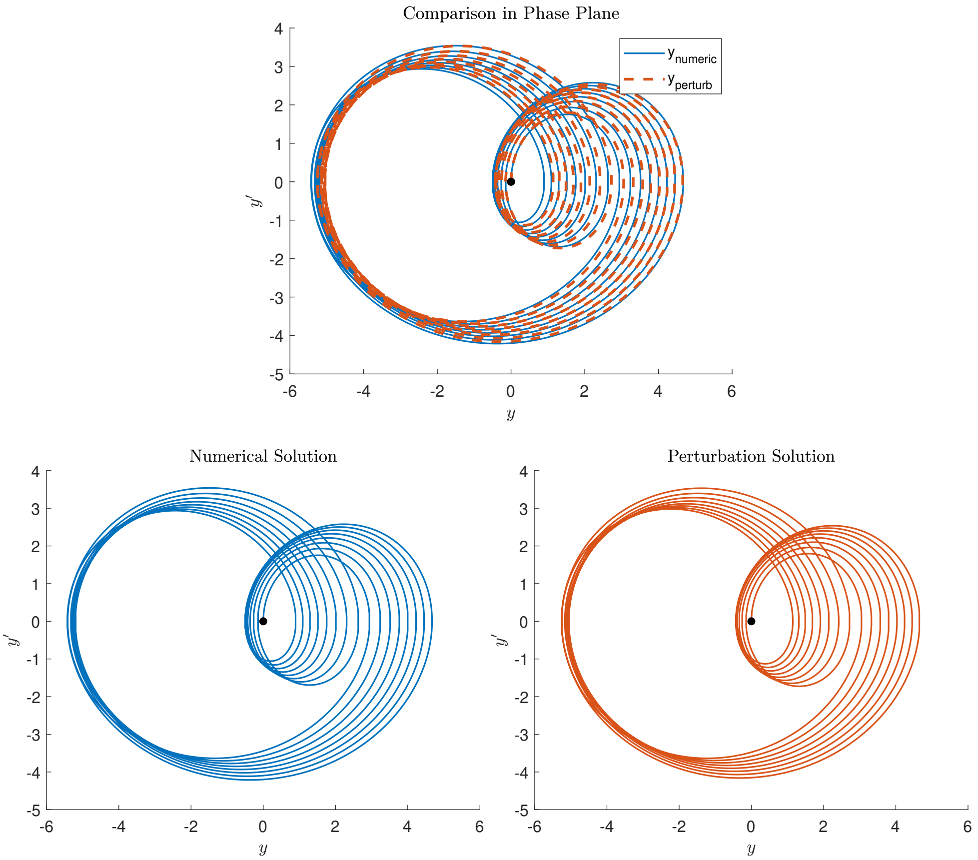

Figure 6. Comparison of numerical solution and approximate solution in the phase plane ($\delta < \gamma$)

The error obtained in the $\delta < \gamma$ case is relatively big compared to the ε chosen and it can also be seen that the error is getting bigger. In the phase plane, there is a difference between the numerical solution and the approximate solution. However, when the solutions are compared side by side, the approximate solution can still explain the behavior of the numerical solution for $t < 1/\varepsilon$. Beyond that, the solution becomes infinite, which no longer represents the model since the position of the plate on the model cannot possibly move towards infinity.

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Case 2: $\delta < \gamma$

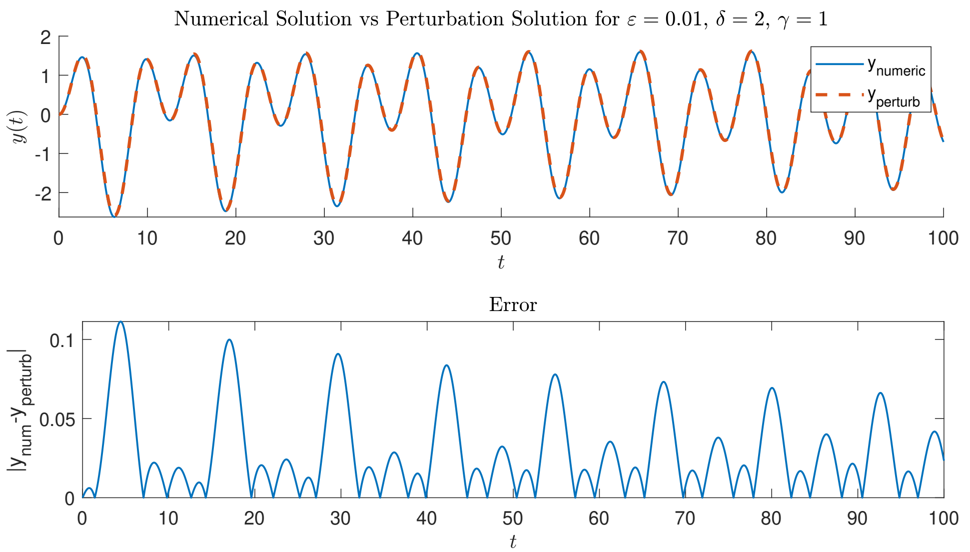

Figure 7. Comparison of numerical solution and approximate solution ($\delta > \gamma$)

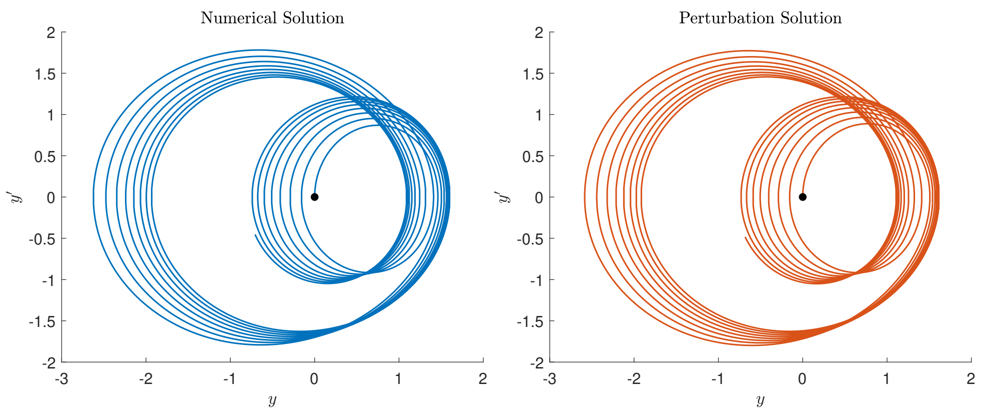

Figure 8. Comparison of numerical solution and approximate solution in the phase plane ($\delta > \gamma$)

It can be seen that the error exceeded 0.1 at the beginning of the simulation. However, as time grows, the error decreases. In the phase plane, there is also no significant difference in the solution.

Case 3: $\delta = \gamma$

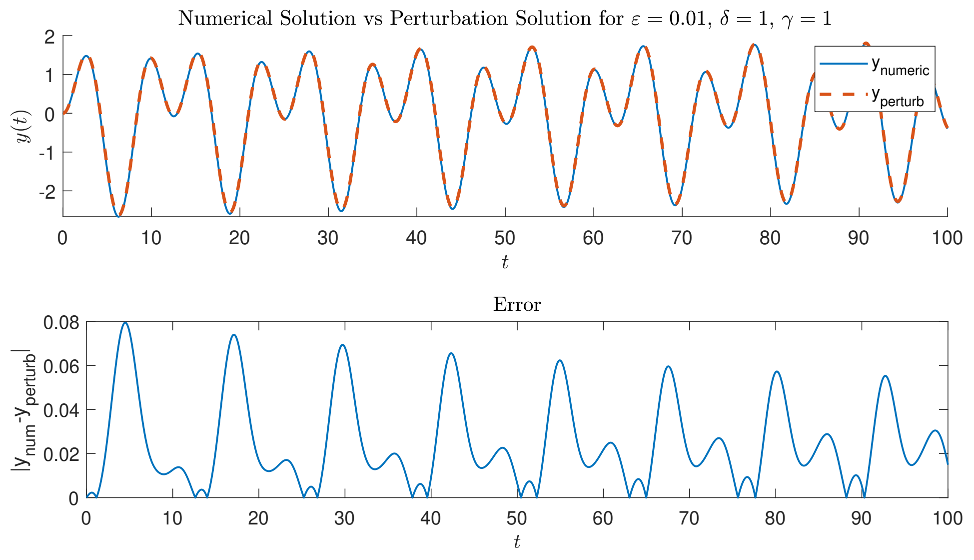

Figure 9. Comparison of numerical solution and approximate solution ($\delta = \gamma$)

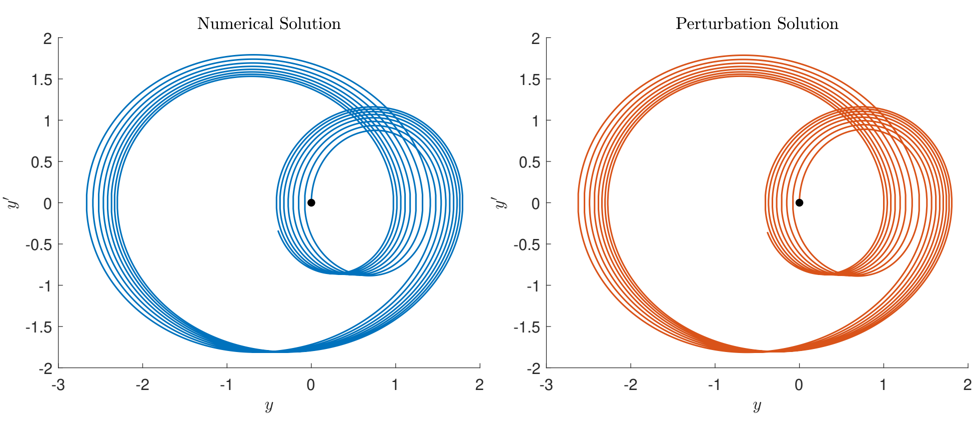

Figure 10. Comparison of numerical solution and approximate solution in the phase plane ($\delta = \gamma$)

The error obtained in this case is observed to be $O(\varepsilon)$ based on numerical simulations, specifically, the error decreases proportionally to ε. As time progresses, the error decreases, confirming the expected scaling behavior. This trend is evident in the numerical results, supporting the conclusion that the error remains of order $O(\varepsilon)$.

4 Conclusion

The conclusions drawn from the analysis indicate that the solution of the system, both with and without damping, remains bounded. In the nonresonance case, the perturbation solution provides a close approximation to the numerical solution, demonstrating its effectiveness. However, in the subharmonic case, the perturbation solution exhibits relatively large errors, especially as ε increases, but it still captures the overall dynamics of the numerical solution. Notably, the perturbation solution remains bounded in all cases except in the subharmonic scenario where $\gamma > \delta$.

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