Effect of Advection on the Modified Schnakenberg System

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Abstract. The Schnakenberg system is a mathematical model that describes the diffusive dynamics of chemical reactions and the formation of space-time patterns. In this paper, we discuss a modified Schnakenberg system including the effects of advection, i.e. reaction diffusion-advection (RDA) system. Stability analysis shows that advection can change the equilibrium state and trigger transitions to complex spatial patterns or oscillatory states. This research uses theoretical analysis methods and literature studies to analyze the impact of diffusion-advection reactions on the modified Schnakenberg system. In this system, only one equilibrium point $S(a+b, \frac{b+(a+b)^3}{c(a+b)^2})$ is found with the stability condition $b-a < (1+c)(a+b)^3$. The results also show that advection in the modified Schnakenberg system affects the steady-state stability and have practical implications for understanding bifurcation phenomena in chemical and biological systems.

Keywords. Schnakenberg system, reaction diffusion-advection, dynamics system

1 Introduction

Schnakenberg system was first introduced by Schnakenberg in 1979 [1]. Schnakenberg systems consist of two dynamic variables that represent the concentrations of two chemical substances, an activator and an inhibitor. These two substances react according to a chemical equation that follows certain laws of kinetics, while diffusing through diffusion in the medium. Diffusion is a process of moving substances from a high-concentration substance to a low-concentration substance [2]. The diffusion process that occurs in chemical reactions will form a diffusion reaction process.

Diffusion reaction is a process in which two or more chemicals diffuse at unequal rates over a surface and react with each other to form stable patterns such as spots and stripes [3]. In the Schnakenberg system, the diffusion reaction serves to control the process between two chemical substances dispersed in the medium space and affects the dynamics of pattern formation. The analysis of the influence of diffusion on the Schnakenberg system has been discussed by many previous studies[4–6]. In addition to diffusion, there is another factor that can affect the dispersion and pattern formation

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in such systems, namely advection. The advection process involves the displacement of substances due to the flow of the medium which can add complexity to the patterns formed.

The addition of advection in the system can form a reaction-diffusion-advection (RDA). Liu *et al.* [7] has applied advection to the plankton community dynamics system. In addition, Ma and Guo [8] also applied advection to the dynamics system of interspecies competition. In Schnakenberg systems, the movement of chemicals can be driven by flow which makes the system dynamics more complex. This flow not only allows patterns to form, but also allows them to move or change shape over time due to the influence of advection. Different from previous studies, which mainly focused on diffusion-driven dynamics, this study investigates how advection affects the stability of equilibrium points and triggers transitions to complex patterns or oscillatory states. This research provides a new perspective on the interplay between diffusion and advection in determining system behavior by extending the analysis to include advection.

This research aims to develop and analyze the stability of the modified Schnakenberg system by including advection terms. The modified Schnakenberg system used in this study is taken from research by You [9] which then by using nondimensional parameters and adding advection terms produces the following system of equations

$$\begin{aligned} u_t &= u_{xx} + \eta_1 u_x + \gamma(a - u + cu^2 v - u^3), \quad (x,t) \in (0,\ell) \times (0,\infty), \\ v_t &= dv_{xx} + \eta_2 v_x + \gamma(b - cu^2 v + u^3), \quad (x,t) \in (0,\ell) \times (0,\infty), \end{aligned} \tag{1}$$

where u and v are activator and inhibitor respectively, a, b, c, γ are positive constants, d and η are diffusion coefficient and advection coefficient respectively. Neumann boundary conditions are used on the Schnakenberg system to ensure that patterns can form naturally without external influences altering the distribution of chemicals at the edges of the domain. Therefore, the system (1) comes with Neumann boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \\ &u_x(0,t) = u_x(\ell,t) = v_x(0,t) = v_x(\ell,t) = 0, \quad t > 0. \end{aligned} \tag{2}$$

2 Research Methods

This research uses the literature review method to review previous studies related to the Schnakenberg system and stability analysis of partial differential equation systems. The research method includes a comprehensive review of related literature, data collection from reliable sources, and systematic data analysis to achieve the research objectives. This research involves modeling a modified Schnakenberg system, by including diffusion-advection. The steps of this research are shown in Figure 1

3 Results and Discussion

3.1 Schnakenberg System

The general formula of Schnakenberg equation was originally formulated by [1]. In this study, a modification of the Schnakenberg system by [9] containing diffusion will be studied with the following equation

$$\begin{aligned} A_t &= D_A A_{xx} + k_1 - k_2 A + A^2 B - k_3 A^3 \\ B_t &= D_B B_{xx} + k_4 - A^2 B + k_3 A^3. \end{aligned} \tag{3}$$

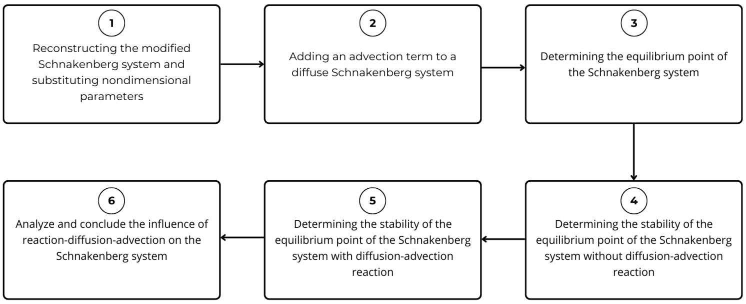

Figure 1. Research steps used in this study.

To analyze the system (3) in detail, non-dimensionality is performed to reduce the system to a simple formulation, which contains fewer free parameters. Defined,

$$A = u\sqrt{\frac{k_2}{k_3}}, \quad B = v\sqrt{\frac{k_2}{k_3}}, \quad t = \frac{t^* L^2}{D_A}, \quad x^* = \frac{x}{L}, \tag{4}$$

$$d = \frac{D_B}{D_A}, \quad a = \frac{k_1}{k_2}\sqrt{\frac{k_3}{k_2}}, \quad b = \frac{k_4}{k_2}\sqrt{\frac{k_3}{k_2}}, \quad \gamma = \frac{L^2 k_2}{D_A}, \quad c = \frac{1}{k_3}. \tag{5}$$

Based on the results of the substitution of the nondimensional parameters (4)-(5) to simplify the notation, the following dimensionless Schnakenberg system is obtained

$$\begin{aligned} u_t =& u_{xx} + \gamma(a - u + cu^2 v - u^3), \quad (x,t) \in (0,\ell) \times (0,\infty), \\ v_t =& dv_{xx} + \gamma(b - cu^2 v + u^3), \quad (x,t) \in (0,\ell) \times (0,\infty), \end{aligned} \tag{6}$$

where a, b, c, γ, d are positive constants. Then, the advection term will be added to the system to study its effect on the stability of the system. So, the formulated system is

$$\begin{aligned} &u_t = u_{xx} + \eta_1 u_x + \gamma(a - u + cu^2 v - u^3), \quad (x,t) \in (0,\ell) \times (0,\infty) \\ &v_t = dv_{xx} + \eta_2 v_x + \gamma(b - cu^2 v + u^3), \quad (x,t) \in (0,\ell) \times (0,\infty) \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \\ &u_x(0,t) = u_x(\ell,t) = v_x(0,t) = v_x(\ell,t) = 0, \quad t > 0. \end{aligned} \tag{7}$$

3.2 Stability analysis without diffusion-advection

This section will ignore the diffusion and advection terms and only consider the ODE equations

$$\begin{aligned} u_t =& \gamma(a - u + cu^2 v - u^3) \\ v_t =& \gamma(b - cu^2 v + u^3), \end{aligned} \tag{8}$$

suppose (u_s, v_s) is the equilibrium point of the system (8), u_s, v_s satisfies

$$\begin{aligned} f(u,v) =& \gamma(a - u + cu^2 v - u^3) = 0, \\ g(u,v) =& \gamma(b - cu^2 v + u^3) = 0. \end{aligned} \tag{9}$$

The modified Schnakenberg system (9), only one equilibrium point given by

$$u_s = a + b, \quad v_s = \frac{b + (a+b)^3}{c(a+b)^2}, \tag{10}$$

Theorem 1. *The system (8) will be stable if it satisfies the stability condition that* $b - a < (1+c)(a+b)^3$.

Proof. The stability of the system (8) with equilibrium point $E = \left(a + b, \frac{b+(a+b)^3}{c(a+b)^2}\right)$ obtains the Jacobi matrix J_E as follows

$$J_E = \begin{pmatrix} (-1 + 2cu_sv_s - 3u_s^2)\gamma & cu_s^2\gamma \\ (-2cu_sv_s + 3u_s^2)\gamma & -cu_s^2\gamma \end{pmatrix}. \tag{11}$$

The determinant and trace of J_E are

$$\det(J_E) = \gamma^2 u_s^2, \quad \operatorname{tr}(J_E) = \gamma(-1 + 2cu_sv_s - (3+c)u_s^2)$$

and at the homogeneous steady states these are

$$\det(J_E) = \gamma^2(a+b)^2, \quad \operatorname{tr}(J_E) = \gamma\Big(-1 + 2\Big(\frac{b + (a+b)^3}{a+b}\Big) - (3+c)(a+b)^2\Big)$$

because a, b, and γ are positive constants it is clear that the determinant is always positive, and therefore, the stability of the equilibrium point to homogeneous disturbance will occur whenever $\operatorname{tr}(J_E) < 0$, i.e. when

$$b - a < (1+c)(a+b)^3$$

□

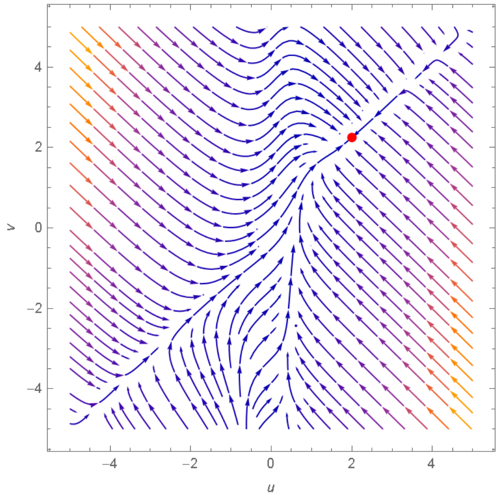

Figure 2. Phase plane at equilibrium point E.

In Figure 2, the phase plane shows that the trajectory around the equilibrium point E is close to that point, so the stability is asymptotically stable if $b - a < (1+c)(a+b)^3$.

This means that when the difference in the values of b and a remains below a critical threshold, the system reaches a stable equilibrium point by maintaining consistent concentrations of activator and inhibitor. If $b - a$ is too large, it means that inhibitor production exceeds activator control, leading to instability. The equilibrium point $E = \left(a + b, \frac{b+(a+b)^3}{c(a+b)^2}\right)$ indicates that the concentration rates of activator and inhibitor go to a certain value.

3.3 Stability analysis with diffusion-advection

To analyze the stability with RDA system, we use the transformation $u(x,t) = w(x,t)e^{\alpha_1 x - \beta_1 t}$ and $v(x,t) = z(x,t)e^{\alpha_2 x - \beta_2 t}$ into (7), gives the following system of equations

$$
\begin{aligned}
w_t &= w_{xx} + \gamma\Big(cw^2 z\mu_1\mu_2 - w^3\mu_1^2 + \frac{a}{\mu_1}\Big),\\
z_t &= dz_{xx} + \gamma\Big(-cw^2 z\mu_1^2 + w^3\frac{\mu_1^3}{\mu_2} + \frac{b}{\mu_2}\Big).
\end{aligned} \tag{12}
$$

with $\mu_1 = e^{-\left(\frac{\eta_1}{2}x + (\frac{\eta_1^2}{4}+\gamma)t\right)}$ and $\mu_2 = e^{-\left(\frac{\eta_2}{2d} + \frac{\eta_2^2}{4d}t\right)}$. Then, to analyze the effect of RDA, the following small pertubation is used

$$
w(x,t) = w_s + \alpha(x,t), \quad z(x,t) = z_s + \beta(x,t).
$$

so that the linearization results of the system (12) are as follows

$$
\begin{aligned}
\alpha_t &= \alpha_{xx} + \gamma(2cw_s z_s\mu_1\mu_2 - 3w_s^2\mu_1^2)\alpha + \gamma c w_s^2\mu_1\mu_2\beta,\\
\beta_t &= d\beta_{xx} - \gamma\Big(2cw_s z_s\mu_1^2 - 3w_s^2\frac{\mu_1^3}{\mu_2}\Big)\alpha - \gamma c w_s^2\mu_1^2\beta.
\end{aligned} \tag{13}
$$

Substitution into the initial transformation $w_s = \frac{u_s}{\mu_1}$ and $z_s = \frac{v_s}{\mu_2}$ gives

$$
\begin{aligned}
\alpha_t &= \alpha_{xx} + \gamma(2cu_s v_s - 3u_s^2)\alpha + \gamma c u_s^2\frac{\mu_2}{\mu_1}\beta,\\
\beta_t &= d\beta_{xx} - \gamma(2cu_s v_s - 3u_s^2)\frac{\mu_1}{\mu_2}\alpha - \gamma c u_s^2\beta.
\end{aligned} \tag{14}
$$

Then, by substituting the pertubation (3.3) into (14) results in a linearized form in the form of (u_s, v_s) as follows

$$
\begin{aligned}
&\alpha_t = \alpha_{xx} + \gamma(2cu_s v_s - 3u_s^2)\alpha + \gamma c u_s^2\frac{\mu_2}{\mu_1}\beta, \quad (x,t) \in (0,\ell)\times(0,\infty),\\
&\beta_t = d\beta_{xx} - \gamma(2cu_s v_s - 3u_s^2)\frac{\mu_1}{\mu_2}\alpha - \gamma c u_s^2\beta, \quad (x,t) \in (0,\ell)\times(0,\infty),\\
&\alpha(x,0) = u_0(x) - u_s, \quad \beta(x,0) = v_0(x) - v_s, \quad x \in \Omega,\\
&\alpha_x(0,t) = \alpha_x(\ell,t) = \beta_x(0,t) = \beta_x(\ell,t) = 0, \quad t > 0.
\end{aligned} \tag{15}
$$

Because the system (7) uses Neumann boundary conditions, the pertubation (3.3) is described as a Fourier series as follows

$$
\alpha(x,t) = \sum_{k\in\mathbb{Z}} \epsilon_k(t)\cos kx, \quad \beta(x,t) = \sum_{k\in\mathbb{Z}} \delta_k(t)\cos kx. \tag{16}
$$

Substituting (16) into the linearized system (15) and removing the common factor of $\cos kx$, for each k gives the system

$$\begin{pmatrix} \dot{\epsilon}_k(t) \\ \dot{\delta}_k(t) \end{pmatrix} = J_k \begin{pmatrix} \epsilon_k(t) \\ \delta_k(t) \end{pmatrix}, \tag{17}$$

with

$$J_k = J - k^2 D, \quad J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \gamma, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \tag{18}$$

and

$$\begin{aligned} a_{11} =& 2cu_s v_s - 3u_s^2, & a_{12} = cu_s^2 \frac{\mu_2}{\mu_1}, \\ a_{21} =& (-2cu_s v_s + 3u_s^2)\frac{\mu_1}{\mu_2}, & a_{22} = -cu_s^2, \end{aligned}$$

so that the resulting J_k matrix is as follows

$$J_k = \begin{pmatrix} \gamma(2cu_s v_s - 3u_s^2) - k^2 & \gamma c u_s^2 \frac{\mu_2}{\mu_1} \\ \gamma(-2cu_s v_s + 3u_s^2)\frac{\mu_1}{\mu_2} & -\gamma c u_s^2 - dk^2 \end{pmatrix} \tag{19}$$

Trace and determinant of matrix J_k are

$$\begin{aligned} \mathrm{tr}(J_k) &= \mathrm{tr}(J_k) - \mathrm{tr}(D)k^2 \\ \det(J_k) &= \det(D)k^4 - (a_{11}d + a_{22})k^2. \end{aligned} \tag{20}$$

Theorem 2. *The system (14) with d is a constant diffusion coefficient, η_1 and η_2 is a constant advection coefficient has an equilibrium point $E = \left(a+b, \frac{b+(a+b)^3}{c(a+b)^2}\right)$ which is stable when:*

(a) $2b < (1+c)(a+b)^3$ *for any* $d > 0$ *and* $k \in \mathbb{Z}$,

(b) $2bd < (c+d)(a+b)^3$ *for any* $d > 0$ *and* $k \in \mathbb{Z} \setminus \{0\}$.

Proof. We simply prove that $\mathrm{tr}(J_k) < 0$ and $\det(J_k) > 0$. Evaluation (20) at the steady state (u_s, v_s) gives

$$\begin{aligned} \mathrm{tr}(J_k) =& -(1+d)k^2 + \gamma\Big(\frac{2b - (1+c)(a+b)^3}{(a+b)}\Big), \\ \det(J_k) =& dk^4 + \gamma\Big(\frac{(c+d)(a+b)^3 - 2bd}{(a+b)}\Big)k^2. \end{aligned} \tag{21}$$

Condition $2b < (1+c)(a+b)^3$ ensure that $\mathrm{tr}(J_k) < 0$ for any $d > 0$ and $k \in \mathbb{Z}$. Then, condition $2bd < (c+d)(a+b)^3$ ensure that $\det(J_k) > 0$ for any $d > 0$ and $k \in \mathbb{Z} \setminus \{0\}$. The first condition ensures stability for all modes including the homogeneous mode, i.e. $k = 0$. The second condition is applying only when considering spatial perturbations, i.e. $k \neq 0$, where diffusion d exists. □

The difference in stability conditions with advection and without advection is shown in Table 1.

Example 1. *Set the parameter values $a = 0.1, c = 0.5,$ and $\gamma = 1$ in the system (15), resulting in the stability region shown in Figure 3.*

Stability condition without advection	Stability condition with advection
$b - a < (1 + c)(a + b)^3$	$2b < (c + 1)(a + b)^3$ for any $d > 0$ and $k \in \mathbb{Z}$ $2bd < (c + d)(a + b)^3$ for any $d > 0$ and $k \in \mathbb{Z} \setminus \{0\}$

Table 1. Stability condition before and after the inclusion of advection.

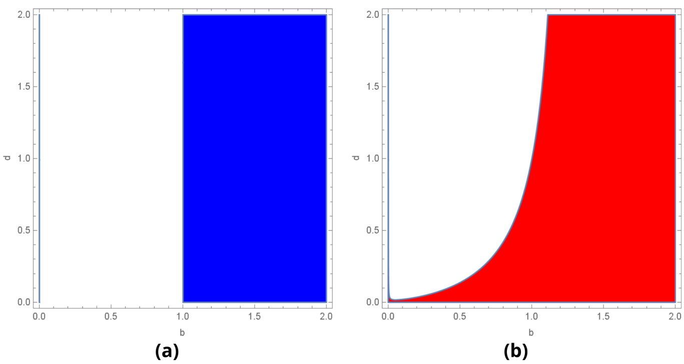

Figure 3. Stability diagram of system (15) in (b, d)-plane for $a = 0.1, c = 0.5$, and $\gamma = 1$. (a) The stability region of system (15) for the condition $2b < (1 + c)(a + b)^3$ for any $d > 0$ and $k \in \mathbb{Z}$. (b) The stability region of system (15) for any $d > 0$ and $k \in \mathbb{Z} \setminus \{0\}$.

4 Conclusions

In this research, it was conclude that the diffusion-advection reaction can affect the stability of equilibrium point $E = \left(a + b, \frac{b+(a+b)^3}{c(a+b)^2}\right)$ in the Schnakenberg system. The equilibrium point which was initially stable when $b - a < (1 + c)(a + b)^3$ becomes stable if $2b < (c + 1)(a + b)^3$ for any $d > 0$ and $k \in \mathbb{Z}$ and $2bd < (c + d)(a + b)^3$ for any $d > 0$ and $k \in \mathbb{Z} \setminus \{0\}$.

Furthermore, this study provides a foundation for understanding bifurcation phenomena in reaction-diffusion-advection systems. The findings could be applied to investigate bifurcation behavior in various systems, such as chemical reactions and biological processes, where transitions between stable and unstable states play a critical role. This connection underscores the relevance of advection in driving complex dynamic behavior and pattern formation, contributing to the broader understanding of bifurcation theory in applied sciences.

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