Dynamics of a Modified Sprott A System

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Abstract. We examine a modified Sprott A system, one of the 17 chaotic systems without equilibria introduced by Jafari, Sprott, and Golpayegani (2013). For specific parameter values, the modified system exhibits invariant spheres. Using a stereographic map, we analyze the stability of the equilibria and demonstrate that all orbits, except for the unstable equilibrium, converge to the stable equilibrium. For other parameter values, the system has neither invariant spheres nor equilibria. Instead, the state space is foliated by tori.

1 Introduction

In 2013, Jafari, Sprott, and Golpayegani [1] (see also [2]) identified seventeen three-dimensional systems of ordinary differential equations with quadratic nonlinearity and without equilibria, which exhibit chaotic behavior. In this paper, we analyze one of those systems, that is, the Sprott A system, which is a special case of the Nosé-Hoover system (see [3–6]). The goal of this paper is to explore the underlying mechanism of its chaotic dynamics. There are two well-known mechanisms for chaos: the cascade of period-doubling bifurcations (see [7]) and the Shilnikov bifurcation (see [8]).

To illustrate the cascade of period-doubling bifurcations, consider a two-dimensional plane transverse to the periodic solution and define the so-called first-return map (or Poincaré map). In this map, the periodic solution appears as a fixed point. By linearizing the map around the fixed point, one observes that as a parameter varies, one of the multipliers crosses 1, leading to the creation of two period two points. These correspond to a periodic solution in the original system with twice the period of the initial periodic solution. The newly formed periodic solution then undergoes a similar bifurcation, resulting in a periodic solution with twice the period again, i.e., four times the period of the original solution. As this process continues, the bifurcation points in parameter space converge to a specific value, and it can be shown that infinitely many unstable periodic solutions exist in the vicinity of this point. Due to their instability, initial conditions are repelled from this neighborhood and cannot settle into any of the periodic solutions, leading to chaotic dynamics (see [7]).

In the Shilnikov bifurcation scenario, chaos arises from the presence of a saddle-type equilibrium with stable and unstable manifolds that intersect *nontransversally* to form a homoclinic orbit. L.P. Shilnikov demonstrated that varying a parameter in such a system can destroy the homoclinic orbit, resulting in the creation of infinitely many periodic solutions (see [8]). Both mechanisms involve bifurcations of equilibria.

Our approach in this paper consists of modifying the original system to allow the existence of an equilibrium. Indeed, the system can be modified in several ways, which might

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differ significantly from one to another. In the original Sprott system we are considering, we recognize the presence of a continuous symmetry for a particular value of the parameter, i.e. it preserves the distance to the origin. Then, our modification is done in such a way that this continuous symmetry (or *integral of motion*) is preserved.

2 The proposed modification

The proposed modification is given by the following system of equations:

$$\begin{aligned} \dot{x} &= y - bxz, \\ \dot{y} &= -x - yz, \\ \dot{z} &= bx^2 + y^2 - a, \end{aligned} \tag{1}$$

where $a, b \geq 0$. If $a = 1$ and $b = 0$, this modified system reduces to the original Sprott A system. In this article, we focus on the case where $b = 1$.

3 Dynamics when $a = 0$

3.1 Invariant structure

3.1.1 Invariant spheres

Define $S(R) := \{(x, y, z) \,|\, x^2 + y^2 + z^2 = R^2, R > 0\}$. It is easy to see that

$$\dot{R} = \frac{d}{dt}(x(t)^2 + y(t)^2 + z(t)^2) = 2(x\dot{x} + y\dot{y} + z\dot{z}) = \frac{-az}{R},$$

which vanishes if $a = 0$, where $(x(t), y(t), z(t))$ is a solution of (1). This means that the sphere $S(R)$ is invariant with respect to the flow of system (1). Thus, the phase space is foliated by these invariant spheres parameterized by the value of the radius R. As a consequence, the dynamics of (1) for $a = 0$ is confined in a two dimensional surface $S(R)$, for fixed $R > 0$. For $R = 0$, the surface becomes the point at the origin.

3.1.2 Manifold of equilibria

It is easy to see that if we set $\dot{x} = 0 = \dot{y} = \dot{z}$, then the $z-$axis:

$$\{(0, 0, z) \,|\, z \in R\}$$

is a line (or manifold) of equilibria. To determine the linear stability of equilibrium, consider the Jacobian matrix at $(0, 0, \tilde{z})$ as follows:

$$\begin{pmatrix} -\tilde{z} & 1 & 0 \\ -1 & -\tilde{z} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_3 = 0, \;\; \lambda_{1,2} = -\tilde{z} \pm i.$$

On a particular sphere, there are two equlibria: one on the upper pole (E_1) and one on the lower pole (E_2). From eigenvalues, E_1 is linearly stable and E_2 is linearly unstable.

3.2 Global stability

Consider the plane $D_0 = \{(x, y, 0) \,|\, x, y \in \mathbb{R}\}$. For $R > 0$ fixed, define a map $\varphi : S(R) \setminus (0, 0, -R) \longrightarrow D_0$ as follows:

$$u := \frac{R}{z+R}x,$$
$$v := \frac{R}{z+R}y.$$

This map is the stereographic map. The origin $(0, 0)$ coresponds to E_1 and the great circle at infinity coresponds to E_2. By differentiating u and v and using $x^2 + y^2 + z^2 = R^2$, we obtain a linear system

$$\begin{aligned} \dot{u} &= -Ru + v, \\ \dot{v} &= -u - Rv, \end{aligned} \tag{2}$$

which have a solution

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{-Rt} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

where $u_0 = u(0)$ and $v_0 = v(0)$. For arbitrary u_0 and v_0, when $t \to \infty$, the orbit converges to $(0, 0)$. This proves the global stability when $a = 0$.

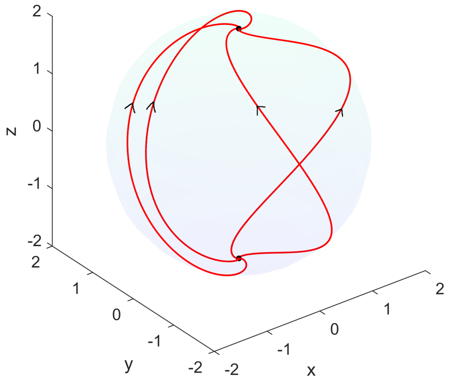

Figure 1. Orbits of system (1) (red) with initial points lie on a sphere with radius $R = 2$ (blue) when $a = 0$.

4 Dynamics when $a > 0$

If $a > 0$, then there are no equilibria and invariant spheres. Consider a transformation

$$\begin{aligned} x(t) &:= r(t)\cos(t + \phi(t)) \text{ with } r > 0, \\ y(t) &:= -r(t)\sin(t + \phi(t)), \\ z(t) &:= z(t). \end{aligned} \tag{3}$$

Substituting this to the system (1), we derive:

$$\begin{pmatrix} \dot{r} \\ \dot{\phi} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -rz \\ 0 \\ r^2 - a \end{pmatrix}, \tag{4}$$

where $a > 0$. The equilibrium $(\sqrt{a}, \phi_0, 0)$ gives

$$x(t) = \sqrt{a}\cos(t + \phi_0),$$
$$y(t) = -\sqrt{a}\sin(t + \phi_0),$$
$$z(t) = 0,$$

which is a periodic solution. Then, we conclude that

$$\frac{r^2}{2} + \frac{z^2}{2} - a\ln r = C, \tag{5}$$

with $C = \frac{r_0^2}{2} + \frac{z_0^2}{2} - a\ln r_0$, is a first integral for the system. For $C > a(1 - \ln a)/2$, (5) is a closed curve as shown in the figure 2. The first integral curve indicates that the orbits lie on tori. Consequently, the phase space is foliated by tori when $a > 0$. Furthermore, the system is integrable which implies that chaotic behavior is not possible. This is in contrast with the original Sprott A system which exhibits chaotic behavior.

Further analysis of the original Sprott system, that is, system (1) with $a = 1$ and $b = 0$, has been done in [9], [10], and [11]. In [9] and [10], the existence of chaotic behavior is proved, while in [11] the analysis is done by means of constructing an approximation to the system by using the averaging method [12, 13]. It is important to note that the mechanism that creates chaos remains unclear in all of these studies. Our modification also are not able to pin point the origin of this chaotic behavior. However, it is important to note that the approximate system constructed in [11] is similar to our modified system.

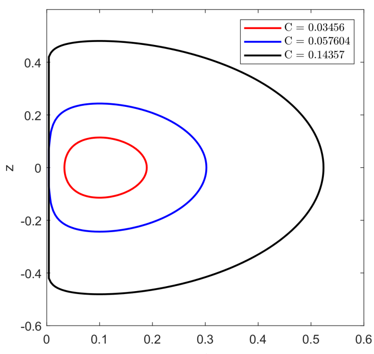

Figure 2. First integral curve (5) for different values of C with $a = 0.01$.

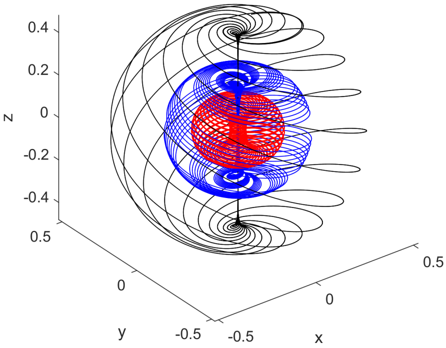

Figure 3. Orbits of system (1) associated to first integral curve in figure (2) with $a = 0.01$. The initial points are determined using the equation (5)

5 Concluding remarks

When $a = 0$, the phase space is foliated by invariant spheres, each containing two equilibria, resembling the dynamics of the original Sprott A system. On each sphere, all orbits, except the unstable equilibrium, converge to the stable equilibrium.

For $a > 0$, the dynamics differ significantly: there are no equilibria, and the phase space is foliated by tori. Unlike the original Sprott A system, no chaotic behavior occurs when $a > 0$, as all orbits lies on a torus.

References

[1] S. Jafari, J.C. Sprott, and M.R.H. Golpayegani, *Elementary Quadratic Chaotic Flows with no Equilibria*. Physics Letters A **377**, 699–702 (2013). https://doi.org/10.1016/j.physleta.2013.01.009

[2] J.C. Sprott, *Some Simple Chaotic Flows*. Phys. Rev. E **50**, R647–R650 (1994). https://doi.org/10.1103/PhysRevE.50.R647.

[3] W.G. Hoover, *Canonical Dynamics: Equilibrium Phasespace Distributions*. Phys. Rev. A **31**, 1695–1697 (1985). https://doi.org/10.1103/PhysRevA.31.1695.

[4] W.G. Hoover, *Remark on 'Some Simple Chaotic Flows'*. Phys. Rev. E **51**, 759–760 (1995). https://doi.org/10.1103/PhysRevE.51.759.

[5] S. Nosé, *A Unified Formulation of the Constant Temperature Molecular-Dynamics Methods*. J. Chem. Phys. **81**, 511–519 (1984). https://doi.org/10.1063/1.447334.

[6] H.A. Posch, W.G. Hoover, F.J. Vesely, *Canonical Dynamics of the Nosé oscillator: Stability, Order, and Chaos*. Phys. Rev. A **33**, 4253–4265 (1986). https://doi.org/10.1103/PhysRevA.33.4253.

[7] J.A. Yorke, K.T. Alligood, *Period-Doubling Cascades of Attractor: A Prerequisite for Horseshoes*. Communications in mathematical physics 101 **3**, 305-321 (1985). https://doi.org/10.1007/BF01216092.

[8] L.P. Shilnikov, *On the Generation of a Periodic Motion from a Trajectory Which Leaves and Re-Enters a Saddle-Saddle State of Equilibrium*. Soviet Math. Dokl. **7** , 1155–1158 (1966).

[9] M. Messias and A.C. Reinol, *On the Formation of Hidden Chaotic Attractors and Nested Invariant Tori in the Sprott A System*. Nonlinear Dyn. **88**, 807–821 (2017). https://DOI10.1007/s11071-016-3277-0.

[10] M. Messias and A.C. Reinol, *On the existence of periodic orbits and KAM tori in the Sprott A system: a special case of the Nosé–Hoover oscillator*. Nonlinear Dyn. **92**, 1287–1297 (2018). https://doi.org/10.1007/s11071-018-4125-1.

[11] F. Verhulst dan T. Bakri, *Time-reversal, Tori Families, and Canards in the Sprott A and NE9 Systems.* Chaos **32**, 083119-1-12 (2022). https://doi.org/10.1063/5.0097508.

[12] J.A. Sanders, F. Verhulst, J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*. Springer, New York (2007).

[13] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*. Springer-Verlag, Berlin (1996).