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Isomorphism of Matrix Algebras over Cuntz Algebras

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Abstract. Starting with a Cuntz algebra O_n constructed by n isometries, we discuss a C^*-algebra consisting of elements of a fixed size k square matrix, where the entries of matrix are from the Cuntz algebra O_n. It is surprising to find that if k divides n, the resulting C^*-algebra of matrix is isomorphic to the Cuntz algebra O_n. We extend this result to cases where k is larger than n, showing that the same conclusion holds provided that every prime factor of k divides n.

1 Introduction

Suppose we have a Hilbert space H over the complex numbers $\mathbb{C}$. The set $B(H)$ of all bounded linear operators on H forms a normed vector space over $\mathbb{C}$. Additionally, we can define a multiplication operation on this set through the composition of two operators. With these two operations, addition and multiplication, the set $B(H)$ acquires an algebraic structure known as a normed algebra.

As an algebra, $B(H)$ doesn't form a field, which is different with the set of complex or real numbers. Nevertheless, similar to forming matrices with real or complex number elements, we can consider matrices whose elements come from a subalgebra in $B(H)$.

We need a structure that is more specific than just a subalgebra, as it also requires an involution operation. In $B(H)$, this involution is represented by the adjoint operator. We will consider a subalgebra constructed from n isometries that is closed under involution. This is known as the Cuntz algebra (see [1]).

The Cuntz algebra has emerged as a fascinating topic in the field of operator algebras, attracting significant attention from researchers in recent years. Several studies have explored its diverse properties and applications, contributing to a deeper understanding of its structure and significance. For instance, [2] investigated the classification of near-group categories using a Cuntz algebra approach, providing a novel perspective on its categorical framework. [3] examined representations of Cuntz algebras associated with random walks on graphs, establishing connections between operator algebras and probability theory. [4, 5] made substantial contributions by studying extension algebras of Cuntz algebras, revealing their intricate algebraic properties and extensions. [6] analyzed the conditions under which the Cuntz–Krieger algebra of a higher-rank graph becomes approximately finite-dimensional, bridging

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graph theory and operator algebras. Additionally, [7] explored symmetry properties in the Cuntz algebra on two generators, while [8] investigated branching laws for endomorphisms of fermions and their relationship with the Cuntz algebra $\mathcal{O}_2$. These studies collectively highlight the versatility and richness of Cuntz algebras, underscoring their importance in both theoretical and applied mathematics.

In this paper, we examine a C^*-algebra constructed from $k \times k$ matrices whose entries are elements of the Cuntz algebra $\mathcal{O}_n$. We show that when k divides n, the resulting algebra exhibits a structure similar to the Cuntz algebra itself. Moreover, we extend this result by demonstrating that the matrix size k can be **larger** than n, provided that all prime factors of k divide n (see Corollary 6). This finding broadens the known constraints on k and refines the relationship between matrix algebras and Cuntz algebras, contrasting with the classical case of matrices over the complex numbers.

2 Discussion

We consider an abstraction of the structure of the set of all bounded linear operators on a Hilbert space H. A set $\mathfrak{B}$ is called an algebra over the complex numbers $\mathbb{C}$ if $\mathfrak{B}$ is a vector space over $\mathbb{C}$, and it is equipped with an associative multiplication operation, i.e., for every $X, Y \in \mathfrak{B}$, then $XY \in \mathfrak{B}$.

As in $B(H)$, the algebra $\mathfrak{B}$ can also be equipped with a norm $\|\cdot\|$, which is a function from $\mathfrak{B}$ to the real numbers satisfying the following properties:

1. **Non-negativity and Definiteness**: For every $X \in \mathfrak{B}$,

$$\|X\| \geq 0, \quad \text{and} \quad \|X\| = 0 \text{ if and only if } X = 0.$$

2. **Scalar Multiplication**: For every $\alpha \in \mathbb{C}$ and $X \in \mathfrak{B}$,

$$\|\alpha X\| = |\alpha|\|X\|.$$

3. **Subadditivity (Triangle Inequality)**: For every $X, Y \in \mathfrak{B}$,

$$\|X + Y\| \leq \|X\| + \|Y\|.$$

4. **Submultiplicativity**: For every $X, Y \in \mathfrak{B}$,

$$\|XY\| \leq \|X\|\|Y\|.$$

If the norm is complete, meaning that every Cauchy sequence in $\mathfrak{B}$ converges, then $\mathfrak{B}$ is called a **Banach algebra**. We also define an **involution** on $\mathfrak{B}$, which is a map $X \mapsto X^*$ for $X \in \mathfrak{B}$. This involution is compatible with the algebraic structure of $\mathfrak{B}$ and satisfies the following properties:

1. **Involution Property**: For every $X \in \mathfrak{B}$,

$$(X^*)^* = X.$$

2. **Antilinearity**: For every $X, Y \in \mathfrak{B}$ and every $\alpha \in \mathbb{C}$,

$$(\alpha X + Y)^* = \overline{\alpha} X^* + Y^*.$$

3. **Multiplication Compatibility**: For every $X, Y \in \mathfrak{B}$,

$$(XY)^* = Y^* X^*.$$

A C^***-algebra** B is defined as a Banach algebra with an involution $*$ that satisfies

$$\|X^* X\| = \|X\|^2 \quad \text{for all } X \in B.$$

If the C^*-algebra has a multiplicative identity element, it is called a **unital** C^***-algebra**.

In the context of operators, we recognize an isometry S in $B(H)$ as an operator that satisfies $\|Sx\| = \|x\|$ for every $x \in H$. This condition can be expressed as $S^*S = I$. However, in infinite dimensions, there exist isometries S such that SS^* is merely a projection. Following Cuntz [1], we construct a subalgebra generated by a set of isometries with specific properties.

Definition 1. *For natural number $n \geq 2$, the* ***Cuntz algebra*** *$\mathcal{O}_n$ is the C^*-algebra generated by isometries $S_1, \ldots, S_n$ satisfying the relations:*

$$\sum_{i=1}^{n} S_i S_i^* = I, \qquad \textit{and} \qquad S_i^* S_j = \delta_{ij} I, \tag{1}$$

where δ_{ij} is the Kronecker delta, and I is the multiplicative identity element.

In some literature ([9–11]), the Cuntz algebra is described as a universal algebra. However, this paper does not directly use that universal property.

The Cuntz algebra $\mathcal{O}_n$ is unique up to isomorphism. Consequently, if a C^*-algebra $\mathfrak{A}$ contains elements $t_1, \ldots, t_n$ that meet the condition in (1), then the *-homomorphism $\varphi : \mathcal{O}_n \to \mathfrak{A}$ that satisfies $\varphi(s_1) = t_1, \ldots, \varphi(s_n) = t_n$ is injective. This implies that the C^*-subalgebra of $\mathfrak{A}$ generated by $t_1, \ldots, t_n$ is isomorphic to $\mathcal{O}_n$.

Next, we construct a C^*-algebra where the elements are matrices whose entries belong to the Cuntz algebra $\mathcal{O}_n$.

Definition 2. *Let k be a natural number. The matrix algebra of k-square matrix over Cuntz Algebra $\mathcal{O}_n$ is defined by the set*

$$M_k(\mathcal{O}_n) = \{T \mid T \textit{ is a } k \times k \textit{ matrix with elements in } \mathcal{O}_n.\}$$

As an example, one of the elements in $M_2(\mathcal{O}_3)$ is

$$T = \begin{bmatrix} S_1 - 2iS_2S_1^* & 4S_2S_1^* \\ S_1 & S_3 + S_1^2 \end{bmatrix}$$

with $\{S_1, S_2, S_3\}$ being the generators of $\mathcal{O}_3$ as in (1). Note that we can express T as follows:

$$T = S_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + (S_2S_1^*) \begin{bmatrix} -2i & 4 \\ 0 & 0 \end{bmatrix} + (S_3 + S_1^2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Each element in the summation can be viewed as $A \otimes B$, which is the tensor product of an element A in $M_2(\mathbb{C})$ (a 2×2 matrix) and B an element in $\mathcal{O}_3$. Using the tensor product notation, the element above can be written as:

$$T = (A_1 \otimes B_1) + (A_2 \otimes B_2) + (A_3 \otimes B_3)$$

with $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -2i & 4 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{C})$ and $B_1 = S_1, B_2 = S_2S_1^*, B_3 = S_3 + S_1^2 \in \mathcal{O}_3$.

In general, we can conclude the following property:

Proposition 3. *Every $T \in M_k(O_n)$ can be expressed as*

$$T = \sum_{i=1}^{N} A_i \otimes B_i, \tag{2}$$

for some positive integer N, where $A_i \in M_k(\mathbb{C})$ and $B_i \in O_n$ for all $i = 1, \ldots, N$.

Proof. Let $T \in M_k(O_n)$. Then there exist elements $t_{ij} \in O_n$ for $1 \leq i, j \leq k$ such that

$$T = \begin{bmatrix} t_{11} & \cdots & t_{1k} \\ \vdots & \ddots & \vdots \\ t_{k1} & \cdots & t_{kk} \end{bmatrix} = \sum_{i=1}^{k} \sum_{j=1}^{k} t_{ij} E_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{k} E_{ij} \otimes t_{ij},$$

where E_{ij} is the elementary matrix that has 1 in the (i, j)-th entry and zeros elsewhere. □

The first step is to prove that $M_k(O_n)$ is also a C^*-algebra. It is straightforward to see that $M_k(O_n)$ forms a vector space over the field $\mathbb{C}$. The multiplication operation in $M_k(O_n)$ is defined using the standard matrix multiplication. For $A, C \in M_k(\mathbb{C})$ and $B, D \in O_n$, the multiplication operation is given by

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$

where AC denotes the usual matrix multiplication, and BD is the product in O_n. Next, for every $A \in M_k(\mathbb{C})$ and $B \in O_n$, the *involution* of an element $T = A \otimes B \in M_k(O_n)$ is defined as

$$T^* = A^* \otimes B^*,$$

where A^* denotes the conjugate transpose of the complex matrix A, and B^* is the involution in O_n. Furthermore, the norm of an element in $M_k(O_n)$ is defined analogously to the operator norm in a Hilbert space, given by

$$\left\| \begin{bmatrix} t_{11} & \cdots & t_{1k} \\ \vdots & \ddots & \vdots \\ t_{k1} & \cdots & t_{kk} \end{bmatrix} \right\| = \left(\sup_{\sum_{j=1}^{k} \|x_j\|^2 = 1} \sum_{i=1}^{k} \left\| \sum_{j=1}^{k} t_{ij} x_j \right\|^2 \right)^{1/2},$$

where $t_{ij}, x_j \in O_n$ for $1 \leq i, j \leq k$ (see [12], Appendix B).

Given that $M_k(O_n)$ is a C^*-algebra, we choose a generating set for it, though not necessarily a minimal one.

Lemma 4. *Let n and k be natural numbers, and let $S_1, S_2, \ldots, S_n$ be the isometries that generate the Cuntz algebra O_n, i.e., the isometries satisfying (1). The set*

$$\{E_{ij} \otimes S_l \mid 1 \leq i, j \leq k,\ 1 \leq l \leq n\} \tag{3}$$

generates the C^*-algebra $M_k(O_n)$.

Proof. Let $\mathfrak{A}$ be the C^*-subalgebra generated by the set (3). Then, for $1 \leq i, j \leq k$ and $1 \leq l \leq n$, we have

$$E_{ij} \otimes S_l^* = (E_{ji} \otimes S_l)^* \in \mathfrak{A}.$$

This implies that

$$E_{ij} \otimes I_{O_n} = \sum_{p=1}^{n} E_{ij} \otimes S_p S_p^* = \sum_{p=1}^{n} (E_{ii} \otimes S_p)(E_{ij} \otimes S_p^*) \in \mathfrak{A}.$$

Meanwhile, since $\{E_{11} \otimes S_l\}_{l=1}^n \subset \mathfrak{A}$ and $\{S_l\}_{l=1}^n$ generates O_n, it follows that for every $t \in O_n$,

$$E_{11} \otimes t \in \mathfrak{A}.$$

Therefore, in view of the proof of Proposition 3, for every $T \in M_k(O_n)$, we have

$$T = \sum_{i=1}^{k} \sum_{j=1}^{k} E_{ij} \otimes t_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{k} (E_{i1} \otimes I_{O_n})(E_{11} \otimes t_{ij})(E_{1j} \otimes I_{O_n}) \in \mathfrak{A}.$$

Thus, the proof is complete. □

Surprisingly, $M_k(O_n)$ is isomorphic to O_n itself as long as k divides n.

Theorem 5. *Let n, k be natural numbers. If k divides n, then $M_k(O_n)$ is isomorphic to O_n.*

Proof. First, we will construct the generator of $M_k(O_n)$. For $0 \leq j < n/k$ and $1 \leq i \leq k$, we define matrices

$$T_{kj+i} = \sum_{l=1}^{k} E_{il} \otimes S_{kj+l}$$

where E_{il} is a matrix unit, and S_{kj+l} is the element in equation (1). It will be shown that

$$\sum_{i=1}^{k} \sum_{j=1}^{n/k} T_{kj+i} T_{kj+i}^* = I.$$

To do this, we compute:

$$\begin{aligned}
\sum_{i=1}^{k} \sum_{j=1}^{n/k} T_{kj+i} T_{kj+i}^* &= \sum_{i=1}^{k} \sum_{j=1}^{n/k} \left(\sum_{l=1}^{k} E_{il} \otimes S_{kj+l} \right) \left(\sum_{l=1}^{k} E_{il} \otimes S_{kj+l} \right)^* \\
&= \sum_{i=1}^{k} \sum_{j=1}^{n/k} \left(\sum_{l=1}^{k} E_{il} \otimes S_{kj+l} \right) \left(\sum_{m=1}^{k} E_{im}^* \otimes S_{kj+m}^* \right) \\
&= \sum_{i=1}^{k} \sum_{j=1}^{n/k} \sum_{l=1}^{k} \sum_{m=1}^{k} (E_{il} E_{im}^*) \otimes \left(S_{kj+l} S_{kj+m}^* \right) \\
&= \sum_{i=1}^{k} E_{ii} \otimes \left(\sum_{j=1}^{n/k} \sum_{l=1}^{k} S_{kj+l} S_{kj+l}^* \right) \\
&= I_{M_k(\mathbb{C})} \otimes I_{O_n} = I_{M_k(O_n)}.
\end{aligned}$$

Next, we show that T_{kj+i} is an isometry. To do this, we compute:

$$\begin{aligned}
T_{kj+i}^* T_{kj+i} &= \left(\sum_{l=1}^{k} E_{il}^* \otimes S_{kj+l}^* \right) \left(\sum_{l=1}^{k} E_{il} \otimes S_{kj+l} \right) \\
&= \sum_{l=1}^{k} E_{ll} \otimes S_{kj+l}^* S_{kj+l} \\
&= \sum_{l=1}^{k} E_{ll} \otimes I_{O_n} = I_{M_k(\mathbb{C})} \otimes I_{O_n} = I_{M_k(O_n)}.
\end{aligned}$$

At this point, we have shown that the C^*-subalgebra of $M_k(O_n)$ generated by $\{T_l\}_{l=1}^n$ is isomorphic to the Cuntz algebra O_n. Now, we only need to show that $\{T_l\}_{l=1}^n$ spans the entire $M_k(O_n)$. To do this, we demonstrate that each element in the generating set (3) can be constructed using $\{T_l\}_{l=1}^n$. The key result needed for this final step is the following expression:

$$\begin{aligned}\sum_{j=1}^{n/k} T_{kj+p}T^*_{kj+q} &= \sum_{j=1}^{n/k}\left(\sum_{l=1}^{k} E_{pl}\otimes S_{kj+l}\right)\left(\sum_{l=1}^{k} E_{ql}\otimes S_{kj+l}\right)^*\\ &= \sum_{j=1}^{n/k}\left(\sum_{l=1}^{k} E_{pl}\otimes S_{kj+l}\right)\left(\sum_{m=1}^{k} E^*_{qm}\otimes S^*_{kj+m}\right)\end{aligned}$$

Using the multiplication operation, we obtain:

$$\begin{aligned}\sum_{j=1}^{n/k} T_{kj+p}T^*_{kj+q} &= \sum_{j=1}^{n/k}\sum_{l=1}^{k}\sum_{m=1}^{k} E_{pl}E_{mq}\otimes (S_{kj+l}S^*_{kj+m})\\ &= \sum_{j=1}^{n/k}\sum_{l=1}^{k} E_{pq}\otimes \left(S_{kj+l}S^*_{kj+l}\right)\\ &= E_{pq}\otimes\left(\sum_{j=1}^{n/k}\sum_{l=1}^{k} S_{kj+l}S^*_{kj+l}\right)\\ &= E_{pq}\otimes I_{O_n}.\end{aligned}$$

Thus, for any $1 \le p, q, i \le k$ and $0 \le m \le n/k$, we have

$$\begin{aligned}E_{pq}\otimes S_{km+i} &= \sum_{l=1}^{k}\left(E_{pi}E_{il}E_{iq}\right)\otimes (I_{O_n}S_{km+l}I_{O_n})\\ &= \left(E_{pi}\otimes I_{O_n}\right)\left(\sum_{l=1}^{k} E_{il}\otimes S_{km+l}\right)\left(E_{iq}\otimes I_{O_n}\right)\\ &= \left(\sum_{j=1}^{n/k} T_{kj+p}T^*_{kj+i}\right)T_{km+i}\left(\sum_{j=1}^{n/k} T_{kj+i}T^*_{kj+q}\right)\end{aligned}$$

and the proof is complete. □

Finally, we can extend the above result. In this case, we allow k to be greater than n. This leads to the following corollary.

Corollary 6. *Let k and n be positive integers such that every prime factor of k divides n. Then $M_k(O_n)$ is isomorphic to O_n.*

Proof. Suppose that $k = p_m p_{m-1}\cdots p_1$ is the prime factorization of k. We observe that

$$M_k(O_n) \cong M_{p_m}(M_{p_{m-1}}(\cdots(M_{p_2}(M_{p_1}(O_n))))).$$

Since p_i divides n for $i = 1, \ldots, m$, it follows from the previous theorem that

$$M_{p_1}(O_n) \cong O_n.$$

Therefore, by induction on m, we obtain the desired result.

□

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