# Vibration analysis of thick composite sandwich cantilever plates with different core materials

*C.W.* Liu and *Taiyan* Kam*

Mechanical Engineering Department, National Yang Ming Chiao Tung University Hsin Chu, Taiwan, Republic of China

**Keywords:** Composite sandwich plate, Shear deformation theory, Ritz method, Free vibration, Natural frequency, Composite materials.

**Abstract.** Because of their light-weight and high stiffness, composite sandwich cantilever plates have been widely used in different industries. In this paper, a method formulated on the basis of the Ritz method and a simple first-order shear deformation theory (SFSDT) is presented to analyze the free vibration of thin and thick rectangular composite sandwich cantilever plates for determining their modal characteristics. In the Ritz method, two independent sets of characteristic functions, namely, polynomials and trigonometric functions, which satisfy the displacement and force conditions at the fixed edge, have been used to construct the eigenvalue problem for extracting the modal characteristics of the plates. For illustration, some existing theoretical and experimental natural frequencies of composite sandwich plates with different core materials have been used to demonstrate the capability of the proposed method in predicting the actual natural frequencies of thin and thick composite sandwich plates.

## 1 Introduction

Cantilever plates have been used to fabricate turbine blades, fighter spoilers, aircraft stabilizers, and missile fins which are subjected to vibration. Because of their importance, many research papers have been published to focus on the free vibration analysis of composite cantilever plates and different methods have been proposed to determine their modal characteristics [1–3]. On the other hand, compared to composite plates, composite sandwich plates can have more attractive properties such as higher specific bending strength/stiffness and less expensive. Therefore, they have the potential to be used to fabricate many vibration-susceptible structures. Regarding the vibration analysis of structural parts, the Ritz method has been widely used for the free vibration analysis of plates. For instance, Kam et al. [4] used Legendre polynomials as characteristic functions in the Ritz method to study the free vibration and sound radiation of elastically restrained stiffened thin composite plates. When the thickness of a plate increases, the effects of through-thickness shear

* Corresponding author: tykam@nycu.edu.tw

deformation on the plate deflection may become too significant to be neglected. Recently, the simple first-order shear deformation theory (SFSDT) has been used for the analysis of relatively thick plates. For instance, Thai and Choi [5] used SFSDT-based methods to analyze various sandwich plates with different boundary and loading conditions. It is obvious that the incorporation of the advantages of the Ritz method and the SFSDT may produce a useful technique for the vibration analysis of composite sandwich plates.

In this paper, a new SFSDT-based Ritz method is presented to study the free vibration of composite sandwich cantilever plates with different core materials. Appropriate characteristic functions are proposed to approximate the deflection components for formulating the vibration of the plates. The theoretical and experimental natural frequencies obtained using different methods are used to verify the accuracy of the proposed method.

## 2 Free vibration formulation

Consider the rectangular symmetrically laminated sandwich plate of size $a$ (length) × $b$ (width) × $h$ (thickness) with the left end clamped as shown in figure 1. The thicknesses of the face sheets and core are $h_f$ and $h_c$, respectively. The fiber angle of the $i^{\text{th}}$ layer is $\theta_i$. Herein, two coordinate systems, namely, reference coordinate system $x$-$y$-$z$ and natural coordinate system $\xi$-$\eta$-$z$ are used to describe the geometry of the plate. Both $x$-$y$ and $\xi$-$\eta$ are located at the mid-plane of the plate with ($0 \le x \le a$, $-b/2 \le y \le b/2$) and ($0 \le \xi \le 1$, $-0.5 \le \eta \le 0.5$), respectively. The relation between the two coordinate systems is $\xi = x/a$ and $\eta = y/b$.

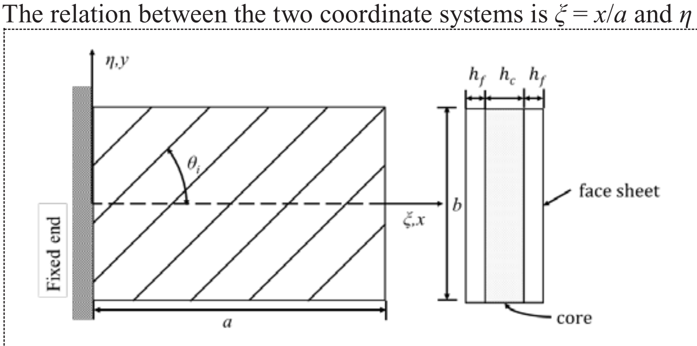


**Fig. 1.** Configuration of composite cantilever plate.

In the SFSDT, the assumptions of plane section remaining plane and no longer perpendicular to the elastic axis after deflection are valid. The total vertical deflection is the sum of two parts, namely, the bending and through-thickness shear-deformation-induced deflections. The displacement field of the symmetrically laminated sandwich plate under free vibration is written as

$$u(x,y,z,t) = -z\frac{\partial w_b}{\partial x}; v(x,y,z,t) = -z\frac{\partial w_b}{\partial y}; w(x,y,z,t) = \left[w_b(x,y,t) + w_s(x,y,t)\right] \tag{1}$$

where $u$, $v$, and $w$ are the displacement components in the $x$-$y$-$z$ coordinate system of the plate; $w_b$ is the vertical deflection induced by bending; and $w_s$ is the vertical deflection induced by through-thickness shear deformation. In the free vibration analysis, the vertical displacement components of the plate in the $\xi$-$\eta$ coordinate system are expressed as

$$w_b(\xi,\eta,t) = W_b(\xi,\eta)\sin\omega t; w_s(\xi,\eta,t) = W_s(\xi,\eta)\sin\omega t \tag{2}$$

where $W_b$ and $W_s$ are the bending and through-thickness shear-deformation-induced deflected shapes, respectively. Discarding the effect of time, the strain-displacement relations of the plate are expressed as

$$\varepsilon_\xi = -\frac{z}{a^2}\frac{\partial^2 W_b}{\partial \xi^2}; \varepsilon_\eta = -\frac{z}{b^2}\frac{\partial^2 W_b}{\partial \eta^2}; \gamma_{\xi\eta} = \frac{\partial v_0}{a\partial\xi} - \frac{2z}{ab}\frac{\partial^2 W_b}{\partial\xi\partial\eta}; \gamma_{\eta z} = \frac{1}{b}\frac{\partial W_s}{\partial \eta}; \gamma_{\xi z} = \frac{1}{a}\frac{\partial W_s}{\partial \xi} \tag{3}$$

where $\varepsilon$ and $\gamma$ are the normal and shear strains, respectively. The stress-strain relations of a composite lamina with arbitrary fiber angle in the face sheet can be expressed in the following general form [6]:

$$\begin{Bmatrix} \sigma_\xi \\ \sigma_\eta \\ \tau_{\eta z} \\ \tau_{\xi z} \\ \tau_{\xi\eta} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & \overline{Q}_{26} \\ 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\xi \\ \varepsilon_\eta \\ \gamma_{\eta z} \\ \gamma_{\xi z} \\ \gamma_{\xi\eta} \end{Bmatrix} \tag{4}$$

where $\sigma$ and $\tau$ are the normal and shear stresses, respectively, and $\overline{Q}_{ij}$ is the transformed lamina stiffness coefficient, which depends on the material properties ($E_1$, $E_2$, $v_{12}$, $G_{23}$, $G_{12}$) and lamina fiber angle. For simplicity, the stress–strain relations in Equation (4) for the $i^{th}$ layer in the plate can be written in matrix form as

$$\sigma_i = \overline{Q}_i \varepsilon_i \tag{5}$$

For the core material with isotropic properties, the fiber angle becomes zero and $E_1=E_2=E$, $v_{12}=v_{21}=v$, $G_{13}=G_{23}=G_{12}=G$, the strain energy $U$ of the plate can be expressed as

$$U = \int \sigma^T{}_i \varepsilon_i dV = \frac{1}{2}\int \varepsilon^T{}_i \overline{Q}_i \varepsilon_i dV \tag{6}$$

Herein, without loss of generality, the shear correction factor is assumed to be 0.8. With the consideration of the rotatory inertia effects, the kinetic energy $T$ of the plate with face sheet parameters of volume $v_f$ and density $\rho_f$ and core parameters of volume $v_c$ and density $\rho_c$ is expressed as

$$T = \frac{1}{2}\int_{v_f} \rho_f \left(u^2 + v^2 + w^2\right) dv_f + \frac{1}{2}\int_{v_c} \rho_c \left(u^2 + v^2 + w^2\right) dv_c \tag{7}$$

The Lagrangian to be used in the Ritz method for the plate is defined as

$$L = T_{\max} - U_{\max} \tag{8}$$

In the Ritz method, each plate displacement can be approximated using a finite series of characteristic functions associated with equal numbers of unknown constants. The extremization of $L$ with respect to the undetermined constants leads to the following eigenvalue problem:

$$[\boldsymbol{K}-\omega^2\boldsymbol{M}]\boldsymbol{C}=0 \tag{9}$$

 https://doi.org/10.1051/itmconf/20257701042

where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{M}$, the mass matrix, and $\boldsymbol{C}$ a vector of the undetermined constants. The solution to the above eigenvalue problem can lead to the determination of the natural frequencies and mode shapes of the plate.

## 3 Ritz method

The characteristic functions for approximating $W_b$ and $W_s$ in the natural coordinate system are chosen, respectively, as

$$W_b = \sum_{i=2}^{M_b} \sum_{j=0}^{N_b} A_{ij} \xi^i \eta^j \tag{10}$$

and

$$W_s = \sum_{i=1}^{M_s} \left\{ B_{i00} \sin\left(\frac{i\pi\xi}{2}\right) \sum_{j=1}^{N_s} \left[ B_{ij1} \sin\left(\frac{i\pi\xi}{2}\right) \sin(j\pi\eta) + B_{ij2} \sin\left(\frac{i\pi\xi}{2}\right) \cos(j\pi\eta) \right] \right\} \tag{11}$$

where $A_{ij}$ and $B_{ijk}$ are unknown constants and $k$ is 0, 1, or 2. The deflection components of equation (10) and equation (11) will then be used to derive equation (9).

## 4 Vibration testing of thick sandwich composite cantilever plate

A rectangular sandwich plate comprising 4 layers of carbon/epoxy laminae at the top and bottom faces and a PU foam core was subjected to free vibration testing. A schematic description of the vibration test apparatus is shown in figure 2. The dimensions of the cantilever composite sandwich plate were $a$ = 112.36 mm, $b$ = 82.01 mm, and $h$ = 25.87 mm, $h_c$=23.91 mm, $h_f$= 0.98 mm with $a/h$ = 4.34. A typical frequency response spectrum is shown in Fig. 3. The measured frequency response spectra were then used to extract the average values of the first six natural frequencies. The average values of the first three natural frequencies ($f_i$ , $i$ = 1-3) determined from all the measured frequency response spectra are $f_1$ = 198.63 Hz, $f_2$ = 558.69 Hz, and $f_3$ = 695.07 Hz.

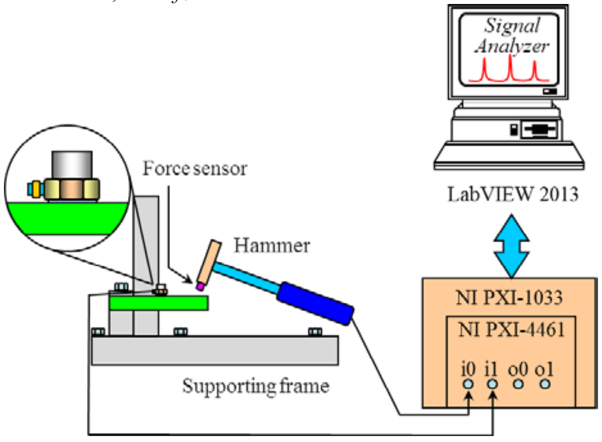


**Fig. 2.** Vibration test apparatus.

It is noted that the material properties of the composite lamina obtained from tests are given as $E_1$ = 54.17 GPa, $E_2$ = 54.17 GPa, $\nu_{12}$ = 0.042, $G_{12}$ = 9.37GPa, $G_{23}$ = 2.7 GPa, $G_{13}$ = 2.7 GPa, and $\rho$ = 1245.00 kg/m$^3$, and the material properties of the foam material are given as $E$ = 1.8 MPa, $\nu$ = 0.3, and $\rho$ = 31 kg/m$^3$.

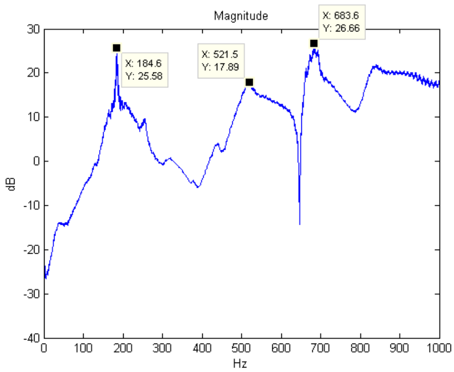


**Fig. 3.** A typical frequency response function of the cantilever composite sandwich plate.

## 5 Results and discussion

The proposed method is used to determine the natural frequencies of a symmetrically laminated graphite/epoxy-aluminum sandwich cantilever plate with $a/h$ = 74.51 in comparison with those obtained using other theoretical and experimental methods. The face layer properties of the graphite/epoxy material are given as follows: $E_1$ = 128 GPa, $E_2$ = 11 GPa, $v_{12}$ = 0.25, $G_{12}$ = 4.48 GPa, $G_{23}$ = 1.53 GPa, $G_{13}$ = 4.48 GPa, $\rho_f$= 1500 kg/m$^3$, Lamina thickness = 0.00013 m

The properties of the 2024-T3 aluminum core are given as follows: $E_c$ = 68.9 GPa, $v_c$ = 0.30, $\rho_c$= 2770 kg/m$^3$, Core thickness = 0.001 m

The natural frequencies of the composite sandwich cantilever plate obtained using different plate theories including Classical Lamination Theory (CLT) and Higher Order Shear Deformation Theory (HOT) are listed in Tables 1 for comparison.

**Table 1.** Natural frequencies of graphite/epoxy-aluminum sandwich [45°, −45°$_2$, 45°, $Al$]$_s$ cantilever plate ($a$=0.152 m, $b$=0.076 m, $a/h$=74.51).

| Method | Natural Frequency (Hz) | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| Experiment [2] | 58.3 | 351.6 | 358 | 1006 | 1113 |
| CLT [2] | 58.5 | 354.7 | 379.6 | 1029.0 | 1187.0 |
| * (error %) | 0.27% | 0.88% | 6.03% | 2.29% | 6.65% |
| HOT [8] | 57.9 | 352.8 | 377.4 | 1020.6 | 1179.2 |
| (error %) | 0.69% | 0.34% | 5.42% | 1.45% | 5.95% |
| Present | 58.4 | 353.7 | 381.0 | 1021.5 | 1186.0 |
| (error %) | 0.10% | 0.60% | 6.42% | 1.54% | 6.56% |

* error % = (|Experimental − Theoretical|/Experimental) × 100%

The inspection of the differences between the experimental and theoretical normalized natural frequencies in Tables 1 shows that the present method can produce results closely matching the experimental data. Next, the proposed method is used to determine the natural frequencies of the composite sandwich cantilever plate which has been tested. The theoretical natural frequencies predicted using the present method is compared with the experimental ones as tabulated in Table 2. It is noted that for comparison purposes, the 3D Solid186 elements of ANSYS [8] have also been used to determine the natural frequencies. The comparison shows that both the present method and ANSYS, no matter whether 3D elements, are able to produce acceptable results. Compared with the experimental natural frequencies, the absolute percentage differences for 3D Solid186, and the present method are less than or equal to 4.17 and 0.59%, respectively.

**Table 2.** Natural frequencies of [0°/PU foam/0°] sandwich composite cantilever plate (*a*/*b*=1.37, *a*/*h*= 4.34).

| Method | Natural Frequency (Hz) | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| Test | 198.63 | 558.69 | 695.07 |
| ANSYS (3D) | 208.41 | 527.98 | 724.03 |
| (error %) | 4.92% | 5.50% | 4.17% |
| Present | 196.41 | 555.37 | 730.32 |
| (error %) | 1.12% | 0.59% | 5.07% |

* error % = (|Test—Theoretical|/Test) × 100%.

## 6 Conclusions

The free vibration of rectangular thick laminated sandwich cantilever plates have been analyzed using a simple shear deformation theory based Ritz method. In the proposed method, two sets of appropriate displacement characteristic functions have been used to formulate the eigenvalue problems of rectangular thick laminated sandwich cantilever plates for natural frequency evaluation via a variational approach. The experimental natural frequencies of a thick composite sandwich plate with PU foam core and a thin one with aluminum core have been used to verifying the accuracy of the theoretical ones. The comparisons between the experimental and theoretical natural frequencies have demonstrated the ability of the present method in producing accurate natural frequencies for both thin and thick composite sandwich cantilever plates comprising different core materials.

This research was funded by National Science and Technology Council of the Republic of China under the grant NSTC 112-2221-E-A49-024.

## References

1. Crawley E F 1979 The natural modes of graphite/epoxy cantilever plates and shells (Journal of Composite Materials vol 13) pp 195-205
2. Narita Y and Iwato N 1990 Vibration studies for symmetrically laminated circular and elliptical plates resting on elastic point supports (Composites Science and Technology vol 39) pp 75-88
3. Ramu I and Mohanty S C 2012 Study on Free Vibration Analysis of Rectangular Plate Structures Using Finite Element Method (Procedia Engineering vol 38) pp 2758-2766
4. Kam T Y Jiang C H and Lee B Y 2012 Vibro-acoustic formulation of elastically restrained shear deformable stiffened rectangular plate (Composite Structures vol 94) pp 3132-3141
5. Thai H T and Choi D H 2013 A simple first-order shear deformation theory for laminated composite plates (Composite Structures vol 106) pp 754-763
6. Tsai S W and Hahn H T 1980 Introduction to composite materials (Technomic Publishing Co., Westport)
7. Nayak A K Sheniio R A and Moy S S J 2004 Transient response of composite sandwich plates (Computers & Structures vol 64) pp 249-267
8. USA 2010 ANSYS 12.1 (ANSYS Inc)